Desirable Outcomes from a Full K-12 Mathematics Program

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Chapter 1

Introduction

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My name is Bert Fristedt. My title is Morse-Alumni Distinguished Teaching Professor of Mathematics at the University of Minnesota, Twin Cities campus. Further information about me is contained in Chapter 19.

This document consists of a representative list of problems that I think that a student graduating from high school and having taken a full K-12 mathematics program should be able to solve. Such students should, in my opinion, be ready to take calculus when they enter college. The problems have not been chosen in order to be particularly challenging or interesting

I am not claiming that a full K-12 mathematics program should be required for either entering college or graduating from high school. Rather, this document focuses on students who plan to go to college and major in a subject that has at least a 2-semester calculus requirement. Some students in accelerated high school programs take calculus in high school. That is fine, provided that the acceleration has not left significant gaps in the understanding of algebra, trigonometry, and geometry. This document does not treat calculus itself, although a couple references to it are given parenthetically.

This document reflects my opinions, and should not be taken as representing an official opinion of the University of Minnesota. However, I have consulted with certain faculty at the Twin Cities campus of the University of Minnesota and have incorporated some of their suggestions.

Some further comments about the list: It is intended to reflect the type of skills, knowledge, and understanding that I view as important outcomes from high school. It does not speak about the path by which these outcomes are to be achieved, except that Chapters 17 and 18 indicate progress that should be attained by the end of sixth grade. Whatever be the path and methods of instruction, the goal should be that each student, as an individual, achieve competence at problems such as those that follow. The phrase 'such as' in the preceding sentence should be interpreted broadly; for every problem in this document there are several other equally important problems that are not represented.

Not all the areas represented in this document are equally important as preparation for taking calculus. Some of the topics, such as probability, that are not crucial for learning calculus are still important as part of a general education, and a high school introduction to them will be of use in a variety of places including some college courses. However, priority must be given to those topics that constitute essential preparation for one or more years of calculus; these are algebra, trigonometry, functions, and aspects of geometry.

The organization is by topic, but I have not attempted to order the problems within each topic in a particular manner. The problems vary considerably in difficulty, but even the most difficult are not designed to test for superior mathematical talent. Descriptions of mathematical topics are often open to different interpretations by different people. Thus, I have chosen a problem list rather than a collection of descriptions to convey my opinion of the mathematics that should be learned by a graduating student from high school who plans to choose a college major requiring at least two semesters of calculus.

Chapter 2

Algebraic Simplifications

Some of the most elementary simplification problems merely involve replacing a variable by a specified number and then performing some arithmetic.

Problem 1 Replace x by 16 in each of the following expressions and then simplify to an exact expression without using a calculator.

(i) $3x - 48$

(ii) $3(x - 48)$

(iii) $\sqrt{x+9}$

(iv) $\sqrt{x} + \sqrt{9}$

(v) $|x - 20| + (x - 20)$

(vi) $\log_{125}(5^x)$

Problem 2 through Problem 16 should be done without a calculator. Notice that doing the last two parts of Problem 3 without a calculator is a hands-on experience at seeing certain patterns.

Problem 2 Simplify the following two expressions:

(i) $3(x - 2) + x[4 - 3(x - 7)]$

(ii) $\frac{(a^3)^5}{(a^7)^2}$

Problem 3 Partially check your answers to the preceding problem by making appropriate calculations using $x = -5$, $x = 3$, $a = 2$, and $a = -2$.

Problem 4 Simplify the following expressions:

(i) $\frac{u-\sqrt{13}}{u^2-13}$

(ii) $\frac{2u^2-7u+5}{4u^2-8u-5}$

(iii) $[3u^{1/2} + 5u^{-1/2}]^2$

Problem 5 For each part of the preceding problem, identify those u for which the given formula is meaningless but for which your simplified formula is meaningful.

Problem 6 Write
$$\frac{5x^4 - 7x^3 + 3x + 6}{2x^2 - 6x + 11}$$
as the sum of a polynomial and a rational function, with the rational function having the property that its numerator has smaller degree than its denominator.

Problem 7 Write the following expression as the quotient of two polynomials:
$$3x^3 - 4x + 3 - \frac{2}{3x^2 + 2}\,.$$

Problem 8 Partially check your answers to the preceding two problems by evaluating both the given expressions and your answers for $x = -2$.

Problem 9 Let b be a number that is larger than 0 and different from 1. Simplify the following expressions:

(i) $\log_b(b^4)$

(ii) $\log_b 5 + \log_b 14$

(iii) $[\log_b 7][\log_7 b^{-3}]$

Problem 10 In this problem e denotes the base of the natural logarithm. Use a calculator to find five-significant-digit approximations of the following numbers.

(i) e^6

(ii) $\log_e 2$

(iii) $(e^{1/2} - e^{-1/2})/2$

(iv) $\log_e(2/7)$

Problem 11 Find the coefficient of $a^7 b^2$ in the expansion of the binomial
$$(a + 3b)^9\,.$$

Problem 12 Write a chain of inequalities that is equivalent to the statement
$$|x - 2| < 7\,,$$
but which does not contain the absolute value symbol.

Problem 13 Let A, B, and C be the following specific sets: $A = \{2, 3, 5, 7\}$, $B = \{3, 5, 8, 13\}$, and $C = \{2, 8, 32\}$. Find:

(i) $A \cap B$

(ii) $(A \cap B) \cap C$

(iii) $(A \cap B) \cup C$

(iv) $(A \cup C) \cap (B \cup C)$

The notation $\{x \colon x > 2\}$ describes the set of those real numbers x which are larger than 2. The alternative notation $\{x \mid x > 2\}$ is used sometimes. In Problem 14 and Problem 15 notation of the type $\{x \colon x > 2\}$ is used rather than notation similar to $\{x \mid x > 2\}$.

Problem 14 Write $\{x \colon |x + 5| \geq 3\}$ as the union of two intervals.

Problem 15 Write $\{x \colon x^2 - 3x + 2 < 0\}$ as one interval if possible. If not, write it either as the empty set or as the union of two intervals, whichever is appropriate.

Problem 16 Rationalize the denominator and simplify:

(i) $\frac{3x+\sqrt{9x^2-5}}{3x-\sqrt{9x^2-5}}$

(ii) $\frac{(x^3-y^3)\,(y-x)}{2xy-\sqrt{2x^4+2y^4}}$

Problem 17 A complex number problem: Write $(3 + 2i)/(5 - 4i)$ in the form $a + bi$ where a and b are real numbers.

Problem 18 Find the square roots of the complex number $3 + 4i$.

Problem 19 through Problem 23 should be done quickly as strictly mental exercises. Rationale: There are many simple simplifications that occur in the middle of larger possibly complex situations. For the purpose of keeping a focus on the overall picture it is very useful to be able to do many simplifications automatically without either pencil or calculator.

Problem 19 Calculate the following numbers:

(i) $25^{3/2}$

(ii) $25^{-3/2}$

(iii) $7^{1/3}7^{5/3}$

Problem 20 Simplify the following expressions, giving your answer orally in an unambiguous manner. For instance, if the answer is $5u^2$ you might say “5 times the square of u”, but if the answer is $(5u)^2$ you might say “the square of $5u$” or “$5u$— quantity squared”.

(i) $6(x + 7) - 3(2x + 10)$

(ii) $x^3 + x^3$

(iii) $\sqrt{27u^2}$

(iv) $(v^2)^5$

(v) $|6x^2 - 3x - 9|/| - 8x^2 + 4x + 12|$

(vi) $(b + a)(b^5 - b^4a + b^3a^2 - b^2a^3 + ba^4 - a^5)$

Problem 21 Simplify the following expressions:

(i) $7\sqrt[3]{2} + \sqrt[3]{54}$

(ii) x^2/x^{-3}

Problem 22 Part (i) Between which two integers does $\sqrt{135}$ lie? Part (ii) What does your answer to Part (i) tell you about $\sqrt{13500}$?

Problem 23 Arithmetic of complex numbers: Evaluate $(\sqrt{3} - 5i)(\sqrt{3} + 5i)$.

In Problem 24 through Problem 29 use a calculator, possibly in conjunction with some paper and pencil calculations, to find the requested approximations.

Problem 24 Find the best five-significant-digit approximation of $\sqrt{7.2}$. Is the square of your approximation accurate through five digits?

Problem 25 Repeat the preceding problem for $\sqrt{72000000}$ and for $\sqrt{72 \times 10^{27}}$.

Problem 26 Replace square root by the $(3/7)^{\text{th}}$ power in the preceding two problems and then do the new problems thus obtained.

Problem 27 Find a six-decimal-place approximation of the sum of the reciprocals of the first 40 positive integers. Organize your work so as to minimize chances for error and the amount of button pushing you have to do or else, if you have a programmable calculator, make use of the programming feature. [Not part of the problem, but those who have had integral calculus can also get approximations by calculating appropriate definite integrals.]

Problem 28 Approximate $\sqrt{2}^{\sqrt{5}} - \sqrt{5}^{\sqrt{2}}$ to five significant figures.

Problem 29 Convert 17 feet, 6.9 inches to the metric system, rounding off to the nearest millimeter.

Chapter 3

Algebraic Equations

When solving equations it is to be understood that all solutions are to be found and that the calculations one makes should show that there are no other solutions; it can happen that there are no solutions in which case a correct response to the problem is to show that there are no solutions. Some books highlight the remarks just made by phrasing problems in the form: Find the solution set of the equation ...; some might even replace the word 'equation' by the term 'open sentence', and the open sentence might be an equation or something else such as an inequality. No calculator should be used except where otherwise stated.

Problem 1 Solve the following equations for v:

(i) $7v - 245 = 15v + 133$

(ii) $6v^2 - 14v + 29 = 3v(5 + 2v)$

Problem 2 This problem concerns the following equation:

$$3v^2 + 14v = -2v + 5\,.$$

(i) Solve it without using a calculator; the symbol $\sqrt{79}$ should appear in your solution.

(ii) Without using a calculator check your answers by inserting them into the given equation.

(iii) Use a calculator and your answer to part (i) to find the best five-decimal-place approximations to your answers.

(iv) Use a calculator to check these approximate answers by inserting them into the given equation, and identify the accuracy that the checking seems to have.

(v) If a graphing calculator is available, use it for an approximate graphical solution as a further check on your decimal approximations.

(vi) If a calculator that actually solves quadratic equations exactly is available, use it for one more check.

Problem 3 For this problem do not use a calculator—exact answers in simple form are required. For which values of the real number k does the following equation for x have two distinct real solutions:

$$5x^2 + kx + 15 = 0\,?$$

For which values of k does the equation have exactly one real solution for x? What is that solution?

Problem 4 Solve each of the following equations for x:

(i) $ax + 10 = -2a + 5x$

(ii) $\log_{10} x = -3$

(iii) $2^x = 9$

(iv) $4^x = 2^{3x-5}$

(v) $\log_b(x-4) + \log_b(x+2) = \log_b(7-4x)$ Be careful!

Problem 5 Use a calculator to find a five-decimal-place approximation to your answer for the third part of the preceding problem.

Problem 6 Find the solution set of each of the following systems of equations in two variables:

(i)

$$\begin{aligned} 3x - 7y &= 2 \\ 6x + 2y &= -7\,. \end{aligned}$$

(ii)

$$\begin{aligned} 3x^2 - 2y^2 + x &= -6 \\ 4x - y &= -2\,. \end{aligned}$$

(iii)

$$\begin{aligned} 3 \cdot 5^x - 2y &= 3 \\ 5^x + 3y &= -4\,. \end{aligned}$$

Problem 7 For each of the following relations involving some or all of x, y, and z, solve for each variable in terms of the others, using more than one formula when necessary.

(i) $(z/x)^y = 4$

(ii) $3x^2 - zx + 5y = 0$

(iii) $\log_3 x + \log_3 y + \log_3 z = 0$

Problem 8 For each part of the preceding problem, some or all of the variables have restrictions on their possible real values in order that (a) all the given expressions be defined and (b) all the formulas obtained be meaningful. Identify all such restrictions, and comment on discrepencies between the restrictions arising from consideration (a) and those arising from consideration (b).

Problem 9 Find all real and complex solutions of the following equations. First express your answers in simplified exact form without using a calculator. Then use a calculator to partially check your answers.

(i) $3x^2 - 7x + 5 = 0$

(ii) $2x^3 - 5x^2 - 3x + 6 = 0$

(iii) $x^4 + 3x^2 - 35 = 0$

Problem 10 Find the solution set of each of the following systems of equations in three variables:

(i)

$$\begin{aligned} 2x - y &= 6 \\ x + y + 3z &= 6 \\ 3x - y - z &= 6\,. \end{aligned}$$

(ii)

$$\begin{aligned} x + y + z &= 3 \\ x - 2y + 4z &= 2\,. \end{aligned}$$

Hint: Express your answer parametrically.

Problem 11 State the Fundamental Theorem of Algebra. Reminder: It concerns polynomial equations.

Formulas that arise from applications often contain several symbols most of which represent quantities that can be different in different problems, but some of which might represent fixed constants. In the following *gravitational attraction formula,*

$$F = G\,\frac{m_1\,m_2}{r^2}\,.$$

G is a universal constant, whereas m_1, m_2, r, and F depend on the setting. The symbol r denotes the distance between two 'point masses' of interest and F the magnitude of the force they exert on each other. The masses themselves are m_1 and m_2, and the *gravitational constant* G is given approximately by

$$G \approx 6.67 \times 10^{-8}\,\frac{\text{centimeters}^3}{\text{grams} \times \text{seconds}^2}\,.$$

The following two problems concern the gravitational attraction formula.

Problem 12 Solve the gravitational attraction formula for r in terms of the other quantities. Also solve it for m_1 in terms of the other quantities.

Problem 13 Fill in the blanks. The gravitational force between two objects varies ________ as the ________ of the ________ between them.

The following problems involve inequalities.

Problem 14 Solve the following inequalities, expressing your answer as an interval, a point, the union of disjoint intervals and/or points, or the empty set, whichever is appropriate.

(i) $7x - 4 \leq 12x + 13$

(ii) $x(4 - x) > 7(x - 4)$

(iii) $(2 + x)(x - 5)^2 \leq 0$

(iv) $3x^2 - 5x + 3 \leq 0$

Problem 15 Without using a graphing calculator sketch the solution set of each of the following inequalities.

(i) $y \leq x^2 - 2$

(ii) $y > e^{0.2x}$

(iii) $xy < 1$

(iv) $|y| \leq -\log_2 |x|$.

Problem 16 Sketch the solution set of each of the following systems, identifying exactly any points of particular interest in your sketch.

(i) $x^2 \leq y \leq 3 - x^2$ [Comment: Part of this problem is to calculate the points where the two parabolas meet; that aspect of the problem is entailed in the final phrase of the above instructions.]

(ii) $y = x^3$, $1 \leq |y| < 8$

(iii) $y^2 < x < \sqrt{|y|}$

(iv) $\arcsin x < y \leq 2x$, $\frac{1}{2} < x \leq 1$

Chapter 4

Algebraic Proofs

The core of an algebraic proof is often manipulative. It is important to insert enough sentences written in good English to enable the person reading the proof to be able to follow the logic.

Problem 1 Prove that for all real numbers x and y,

$$2xy \leq x^2 + y^2 .$$

Problem 2 Let a, b, c, and d be positive integers and suppose that

$$\frac{a}{b} < \frac{c}{d} .$$

Prove that

$$\frac{a}{b} < \frac{a+c}{b+d} .$$

Problem 3 Examine your solution to the preceding problem and decide if your proof used the fact:

(i) that a, b, c, and d were assumed to be integers;

(ii) that b and d were assumed to be positive;

(iii) that a and c were assumed to be positive.

Problem 4 Suppose that a certain examination has two parts, and that on both parts each problem is graded as correct or incorrect, there being no partial credit.

(i) Suppose that a certain student has the same percentage of correct answers on the two parts. Prove that this percentage is also the percentage of correct answers on the entire test by this student.

(ii) Suppose that another student has a higher percentage of correct answers on the second part than on the first part. Use Problem 2 to prove that the percentage of correct answers on the entire examination by this student is larger than her percentage of correct answers on the first part of the examination.

Chapter 5

Exponents in Applications

Compound interest is a mostly mathematical topic which is closely related to laws of exponents and is important for all, even those whose religious beliefs are inconsistent with the charging of interest. It is important for them because interest-type calculations can also be used to evaluate other types of financial endeavors. There are some technical details that are best left to an economics course. For instance, is the interest earned on a given investment the same for each of the four quarters of a year even though the quarters have slightly different numbers of days? Issues such as these are to be ignored in the following problems; so, for instance, in Problem 3 it is to be assumed that the interest rate per quarter is 1/4 of the annual rate even though, in a non-leap-year, the first quarter is shorter than each of the other three quarters.

A calculator for handling the arithmetic should be used on the following problems, but one that has interest- or investment-type capabilities for which the user only need enter data should not be used.

Problem 1 After three years, what is the value to the nearest cent of an initial investment of \$1000 if the interest rate is 6% compounded annually? As a partial check, confirm that your answer is larger than \$1180. Why should one expect an answer that is larger than \$1180?

Problem 2 At 8% compounded annually, how many years (as a whole number of years) does it take to triple the value of an initial investment?

Problem 3 An investment of \$3000 is made at the beginning of the year 2006 at an annual interest rate of 5% compounded quarterly. What is the value of this investment at the end of the first quarter of the year 2012?

The simplest population models are based on exponential functions.

Problem 4 Suppose that at time t days after some fixed starting time, the number of bacteria of type I present is $Ae^{\alpha t}$ and the number of type II is $Be^{\beta t}$, where e is the base of the natural logarithm and A, B, α, and β are positive constants. Suppose that $A < B$ and $\alpha > \beta$. Find a formula for the time at which the numbers of the bacteria of the two types will be equal.

The next problem illustrates another area in which exponential functions arise.

Problem 5 The amount of a radioactive substance in an object decreases exponentially over time, provided that there is no source creating that substance. Thus the formula Ae^{-kt} describes the amount of the substance present t years after a stating time at which there were A grams present. Denote by t_0 the *half-life* of the substance—that is, the time necessary for the amount of the substance become half of what it was at time 0. Find a formula for k in terms of t_0.

Chapter 6

Plane Geometry

All the problems in this chapter concern Euclidean plane geometry, but they are appropriate whether an axiomatic system similar to that of Euclid has been used in learning geometry or, alternatively, coordinates were central to development.

Problem 1 State the Pythagorean Theorem. Make sure you say what all the symbols mean.

Problem 2 Is the triangle with side lengths 20, 21, and 29 a right triangle? Is your answer a direct consequence of the Pythagorean Theorem which you have given in response to the preceding problem?

Problem 3 Calculator problem: Suppose that the two legs of a certain right triangle have lengths equal to 73 feet 0 inches and 94 feet 5 inches. Find the length of the hypotenuse of the triangle to the nearest inch and the measures of the vertex angles to the nearest 10 minutes.

Problem 4 Find the sum of the angular measures at the vertices of a regular n-sided polygon. Does your answer depend on the polygon being regular?

Problem 5 Prove that the perpendicular bisectors of the sides of any triangle meet in a common point which is equidistant from the three vertices of that triangle. [Comment: If you use coordinates in your proof you may choose to locate the coordinate axes so as to simplify the calculations.] [Note: The common point mentioned in this problem is the *circumcenter* of the triangle.]

Problem 6 For which triangles does the circumcenter lie outside the triangle?

Problem 7 Which of the following is not an abbreviation of a criterion for two triangles to be congruent: side-side-side, side-angle-side, side-side-angle, angle-side-angle, angle-angle-side?

Problem 8 Illustrate your answer to the preceding problem by drawing pictures of two triangles that are not congruent but whose relevant parts are congruent.

Problem 9 Use congruent triangles to prove that the diagonals of a rhombus are perpendicular to each other. You may use the theorem that the diagonals of a parallelogram bisect each other (and other more basic facts). Either a two-column format or a paragraph format is acceptable for your proof.

Problem 10 Describe the steps, labeling the steps 1, 2, and so forth, for constructing the perpendicular bisector of a given line segment using straightedge and compass.

Problem 11 To where does the point $(x, y) = (2, -3)$ go if it is first reflected in the line $x = -1$ and then translated 4 units in the positive x-direction?

Problem 12 Define 'trapezoid', and then compare your definition with one in an ordinary English dictionary. A trapezoid is sometimes said to have 'two bases'. Define 'base' in this context. Prove that the area of a trapezoid is the product of its height and the average of the lengths of its two bases, using known formulas for areas of simpler figures. Make sure your proof covers all possible cases.

Problem 13 State the 'parallel postulate' of Euclid, but it is fine if you give an equivalent version of the postulate rather than the version that Euclid stated. [This is an easy problem for those who have had an axiomatic treatment of Euclidean geometry, and is also something that those who have had an analytic approach should know.]

Problem 14 Let C denote the center of a circle and let A, B, and D denote three distinct points on the circle itself such that A, B, and C are not collinear. Suppose that C is in the interior of the angle $\angle ADB$ and that D is in the exterior of $\angle ACB$. Draw a sketch illustrating this given information. Then state and prove a simple relation between the measures of $\angle ADB$ and $\angle ACB$.

Chapter 7

Algebra Used for Geometry

All aspects of all problems in this chapter should be done without a calculator, even the relevant graph sketching. However, it can then be instructive to use a graphing calculator to check your solutions. Graph sketching without a calculator, even if quite inaccurate, is a very hands-on experience that can help in understanding general concepts.

Problem 1 Find an equation of the line passing through the points $(2, 3)$ and $(5, -2)$. Then find where this line intersects each of the coordinate axes.

Problem 2 Find an equation of the line passing through the points $(-3, -3)$ and $(-3, 3)$.

Problem 3 Find the slope of the line given by the equation $3x + 5y = 2$.

Problem 4 This problem concerns the triangle having one vertex at $(0, 0)$ and the other two vertices at the intersections of the line $5x - 12y = -3$ with the coordinate axes.

(i) Find the coordinates of the two vertices of the triangle that lie on the line $5x - 12y = -3$.

(ii) Use the answer to part (i) in order to find the area of the triangle.

(iii) Use the answer to part (ii) and an appropriate calculation in order to find the distance from $(0, 0)$ to the line $5x - 12y = -3$.

Problem 5 For which values of k (if any) are the lines $2x - ky = 3$ and $-3kx + 4y = 0$ parallel? For which values (if any) are they actually the same line? For all values of k for which the two lines meet in exactly one point find that point (where, of course, your formula for that point might involve k).

Problem 6 Find the equation of the circle of radius 5 centered at $(-3, 9)$. Use that equation and related inequalities to decide for each of the following points whether it lies on the circle, inside the circle, or outside the circle: $(0, 0)$, $(-2, 7)$, $(-6, 5)$.

Problem 7 Consider the circle $x^2 + y^2 + 5x - 6y + 10 = 0$.

(i) Find the center and radius of the circle.

(ii) Find all points of intersection (if any) of the circle with the line $2x - y = -3$.

(iii) Sketch a rough graph showing any points of intersection.

Problem 8 Use coordinates to prove that the diagonals of any rectangle have equal length. [Comment: You may orient the coordinate axes so as to simplify the calculation.]

Problem 9 Find the equations of all lines (if there are any) which are tangent to the circle $x^2 + y^2 = 3$ and pass through the point $(0, b)$. *Hint:* For certain values of b there are no such lines—an aspect of solving this problem is to identify those b's.

Problem 10 Find the equation of the set of points equidistant from the line $x = -2$ and the point $(-5, 0)$. Do this problem from basic formulas involving distance, not from having memorized the relation between this setting and certain types of graphs.

Problem 11 Sketch a picture of the solution set of the following system of inequalities:

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ \tfrac{1}{5}x + \tfrac{1}{3}y &\leq 1 \\ x + y &\leq 4 \\ y &\leq x + 1\,. \end{aligned}$$

Then treat the solution set as the domain of the function $3x + 2y$ and find the maximum value of this function.

The above problems integrate algebra with coordinate geometry. But algebra also meshes nicely with many geometric problems having nothing to do with coordinates. Here are some problems of this type.

Problem 12 Let D and E be points on the sides BC and CA, respectively of a triangle $\triangle ABC$. Suppose that the distance from C to E equals 98 and the distance from B to D equals 200. Suppose further that DE is parallel to AB and that the distance from A to E equals the distance from C to D. Draw a sketch illustrating this information and then calculate the exact distance from B to C. [The word 'side' in the first sentence of this paragraph refers to the line segment between two vertices, not the entire line through two vertices. Various books treat this terminology issue differently.]

Problem 13 For a triangle $\triangle ABC$, let a, b, and c denote the lengths of the sides opposite the vertices A, B, and C, respectively. Suppose that $a = 8$, $b = 7$, and $c = 5$. Calculate the exact length of the altitude from A.

Chapter 8

English into Mathematics

In applying mathematics it is important to be able to translate a situation described in English (or some other language) into mathematics. Then one tries to do some mathematical calculations and reasoning in order to say something of interest for the situation as originally described.

The skill in translating back and forth between English and mathematics should be distinguished from another skill important in applying mathematics—namely that of deciding what aspects of the applied setting can be ignored without affecting the validity of any conclusions that are drawn. For instance, when studying the motion of a hard-hit baseball, one needs to decide whether the effect of air is significant. Issues such as this, which can be very complicated, belong primarily to subjects other than mathematics. Accordingly, the somewhat difficult topic of translating English into mathematics is often best learned in situations not requiring significant expertise outside of mathematics.

Problem 1 Susan is planning to buy pencils costing 34 cents each and pens costing \$1.61 each. Write an expression, in dollars, for the total cost in terms of the number x of pencils and the number y of pens that she buys. Then use this expression to calculate the maximum number of pencils she can buy if she is willing to spend up to \$10 and she wants to buy five pens.

Problem 2 Osman has the same plans as Susan, described in the preceding problem, but he lives in a state with a 6% sales tax. Write an expression for his total cost in terms of x and y, and then, as for Susan, calculate the maximum number of pencils he can buy if he is willing to spend up to \$10 and he wants to buy five pens.

Problem 3 Allison and Antonia celebrate their birthdays on the same day each year. Allison is 16 years older than Antonia. How old (in years and, if needed, a fraction of a year) will Allison become on the day that the age of Antonia becomes 5/7 of that of Allison?

Problem 4 A certain square has the same area as a rectangle whose length is 3 more than its width. Find an algebraic expression for the side length of the square in terms of the width of the rectangle. Make sure to identify the meaning of any variable you introduce. Now suppose the side of such a square has length 7. What are the dimensions of the corresponding rectangle?

Problem 5 Find the largest rectangular area that can be enclosed by a long wall on one side and a total of 50 meters of fencing on the other three sides. Is this area more than, less than, or the same as the area that could be enclosed by using the wall as a diameter of a circle and using the fencing to make a semicircle on one side of the diameter?—at the very last step of answering this last question a calculator is useful. [Comment: For the first part of this problem, calculus could be used by those who know it, but it is just as fast to do it without calculus as to do it with calculus.]

Chapter 9

Speed and Distance

The relation between speed and distance traveled is important for everyone's education, even those not planning on work in science. Once the basic relation is understood, then most of the work in solving problems is typically mathematical. Thus it is appropriate that such problems appear in every high school mathematics curriculum.

Problem 1 Felecia traveled from point A to point B at 50 miles per hour, and she arrived at B 90 minutes after she started from A. What is the distance between the points A and B?

Problem 2 Jeremiah traveled one direction on a certain road at 40 miles per hour. Then he traveled 60 miles per hour on the return trip. What was his average speed for the round-trip journey? *Hint:* The answer does not depend on the distance traveled, but part of the problem is to make calculations that show this fact. However, as an introduction to the problem you might want to temporarily assume that the one-way distance is 120 miles.

Problem 3 Eliza and Cecelia left point A at the same time, traveling on separate straight roads making an angle of measure 60° at A. Each traveled at 60 kilometers per hour. What was their distance apart (as the crow flies) 3 hours after they began their journeys?

The preceding problems treat situations involving constant speeds. The motion of an object relatively near the surface of the earth is approximately governed by the following relations:

$$h = -(g/2)t^2 + at + b\,,$$
$$v = -gt + a\,,$$

where t denotes time, h the distance of the object above the surface of the earth, and v the velocity of the object away from the earth (it being negative if, in fact, the object is approaching the earth—the word 'velocity' is usually used in place of 'speed' when one wants to take account of direction of motion; thus 'speed' is the absolute value of 'velocity'). Also, a and b are two constants depending on the specific problem being studied and g is a constant which is approximately equal to 32 feet/seconds^2 [The above formulas are related to the discussion of gravitation near the end of Chapter 3, but the following problem can be done without reference to that relation.]

Problem 4 At time 0 an object leaves a platform 200 feet above the ground at a speed of 50 feet per second directly upwards.

(i) Find the values of the constants a, and b.

(ii) Calculate the number of seconds that elapse until the object hits the ground.

(iii) In doing the preceding part, a negative solution for time was probably obtained and then discarded. Give an interpretation of that negative solution.

(iv) Calculate the speed with which the object hits the ground.

Chapter 10

Trigonometric Simplification

It is important to memorize several trigonometric identities so that they are readily available for use.

Problem 1 Simplify the following expressions:

(i) $\sin\theta \sin 2\theta + \cos\theta \cos 2\theta$

(ii) $(\sec\varphi - 1)(\sec\varphi + 1)$

(iii) $20\cos\psi + (1 - 5\cos\psi)^2$ [Write answer in factored form.]

Problem 2 Given: θ is an angle whose measure is between 90° and 180°, and $\sin\theta = 3/\sqrt{17}$. Without using a calculator find exact expressions for $\cos\theta$, $\tan\theta$, $\cot\theta$, $\sec\theta$, and $\csc\theta$.

Problem 3 Use a calculator to find a six-decimal-place approximation of $\theta \in [0, 2\pi)$ if $\tan\theta < 0$ and 0.35657843 is an eight-decimal-place approximation of $\cos\theta$.

Problem 4 Use the double-angle formula $\cos 2\theta = 1 - 2\sin^2\theta$ to obtain a formula for $\sin(\varphi/2)$ in terms of some trigonometric function evaluated at φ. If you take a square root in your calculation, decide whether it is necessary to consider both the positive square root and the negative square root.

Problem 5 Use the formulas

$$\sin(\alpha + \beta) = \sin\alpha \cos\beta + \sin\beta \cos\alpha$$

and

$$\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta$$

to obtain a formula for $\tan(\alpha + \beta)$ in terms of $\tan\alpha$ and $\tan\beta$.

Problem 6 For the formula you obtained in the preceding problem, discuss how it might be interpreted in case $\tan\alpha$ or $\tan\beta$ is undefined. Also discuss its interpretation in case the denominator in your formula equals 0.

Problem 7 In this problem arcsin denotes the usual choice for the 'inverse' of sin, and similarly for arccos, arctan, and arccot. Also, angular measure is in radians. Simplify the following expressions:

(i) $\arctan x + \text{arccot } x$

(ii) $\arcsin(\sin(2\pi/3))$

(iii) $\cos(\arccos(1/2))$

(iv) $\sin(\arccos(-1/2))$

Problem 8 Find a simple formula for

$$\frac{\cos\varphi}{1-\sin\varphi} + \frac{1-\sin\varphi}{\cos\varphi}$$

involving $\sec\varphi$.

Problem 9 A rectangle is inscribed in a circle of radius 3. Find a formula for the area of that rectangle in terms of the measure of the angle at which its two diagonals meet.

Chapter 11

Trigonometric Equations

Some trigonometric equations have no direct connection with geometry, at least as stated. They are merely equations involving trigonometric functions. Reminder: Such equations typically have infinitely many solutions.

Problem 1 Solve the following equations in radians:

(i) $\cos\theta = \sin 2\theta$

(ii) $\cos\varphi = \sin^2\varphi$

Problem 2 Solve the equation $2\sin^2\omega + 1 = 3\cos\omega$. Express your answers using the arcsin and/or arccos functions, but also use a calculator to obtain nearest-minute approximations of the answers. Check your answers on a graphing calculator by graphing both sides of the given equation.

Problem 3 For each real number k, decide how many solutions the equation $2\sin^2\omega + 1 = k\cos\omega$ has in the interval $[0, 2\pi)$. Here ω denotes radian measure.

The following problems integrate geometry with trigonometry. Even if a relevant trigonometric equation has infinitely many solutions, the related geometry problems does not.

Problem 4 Find the measures of the angles of a triangle whose side lengths are 7, 10, and 14. Write your solutions using exact formulas until the last step and then use a calculator to give approximate values in degrees to the nearest degree.

Problem 5 A parallelogram has sides of length 5 and 8 meeting at an angle of measure 1.3 radians. Find the area of the parallelogram and also the lengths of its two diagonals. Three-decimal-place approximations are requested, for which a calculator may be used.

Problem 6 For an arbitrary regular pentagon, find a formula for the side length x in terms of the diagonal length d. Then use a calculator to approximate any trigonometric functions in your answer. [Comment: For squares and regular pentagons there is only one candidate for the name 'diagonal length'; for regular hexagons, more precision in the statement of the problem would be required.]

Chapter 12

Functions

It is important to have a mental picture of the graphs of several functions; for instance: e^x, $\log_e x$, $\sqrt{x}$, $1/x$, $\tan x$, $\arcsin x$. Then one can use this knowledge to quickly obtain the graphs of related functions.

Problem 1 Without using a calculator, make rough sketches of the graphs of the following functions.

(i) $y = \sqrt{x+3}$

(ii) $y = 3\log_e(x-5)+2$

Problem 2 Without using a calculator, make rough sketches of the graphs of the following functions:

(i) $f(x) = x^2$

(ii) $g(x) = x^3 - 4x$

(iii) $h(x) = 2^{x/3}$.

(iv) $k(x) = \sin 4x$

(v) $G(x) = \cot(1/x)$

(vi) $H(x) = \frac{1}{\pi}\left(\frac{\pi}{2} + \arctan x\right)$

Problem 3 Define three functions by the following formulas:

$$f(x) = x^2 - 3\,,$$
$$g(x) = \sqrt{x+5}\,,$$
$$h(x) = 2x+5$$

—it being understood that in each the case the domain consists of all real numbers x for which the formula is meaningful. Find formulas for each of the following functions, and as part of any answer for which the domain does not contain all x for which the formula is meaningful, identify the domain explicitly.

(i) $f \circ h$

(ii) $g \circ f$

(iii) $f \circ g$

(iv) the inverse function of h

(v) the inverse function of g.

Problem 4 Suppose f and g are two functions related by the formula $g(x) = f(x+2)+3$. Describe what one should do starting with the graph of f to obtain the graph of g.

Problem 5 For each of the following functions, describe its domain and find those points in the domain where the function is positive.

(i) $f(x) = -5(x-2)^2(x+3)(x-5)$

(ii) $g(x) = \sqrt{f(x)}$, for f as defined in the preceding part

(iii) $\log_{10}(3-x) + \log_{10}(3+x)$

(iv) $\arccos[(x+2)/(x+1)]$

Problem 6 Let

$$f(x) = \frac{x^2-4}{x^2-1}\,.$$

Find:

(i) the values of x for which $f(x) = 0$;

(ii) the vertical asymptotes of the graph of f ;

(iii) the values of x for which $f(x) = 2$;

(iv) the horizontal asymptotes of the graph of f;

(v) the intervals on which f is a one-to-one function and for each such interval a formula for the inverse of f restricted to that interval. Be careful to specify the domains of such an inverse function if the domain does not include all values for which the corresponding formula is meaningful.

Problem 7 Find two functions the union of whose graphs is the circle $x^2+y^2 = r^2$. Here r is a fixed but arbitrary positive number.

Problem 8 Find the inverse of the function g defined by

$$g(x) = \arcsin\frac{x}{x+1}\,.$$

Explicitly identify the domain of the inverse function.

Problem 9 Let $f(x) = \cos x$ and $g(x) = \cos 3x$. Find a cubic polynomial p such that g equals the composition $p \circ f$. [This problem is a medium-level trigonometric-identity problem, but the formulation in terms of the composition of functions might make the problem more difficult for some.]

Problem 10 Without doing any writing, give an oral description of the domain and image (sometimes called range) of the functions arcsin, arccos, and arctan. Also describe how you arrived at your answer using your knowledge of the functions sin, cos and tan.

Problem 11 Find those real numbers x for which

$$|\sin x| = -\sin|x|\,.$$

Problem 12 For the function $f(x) = x^3$, simplify the expression

$$\frac{f(x+h)-f(x)}{h}\,.$$

Chapter 13

Solid Geometry

The term ‘solid geometry’ refers to geometry in three dimensions.

Problem 1 Find a formula which relates the length of a long diagonal of a cube to the side length of the cube. [A *long diagonal* connects two vertices and does not lie in any face of the cube.

Problem 2 If one doubles the length of each side of a cube to obtain a new cube, by what do the volume of the cube and the surface area of the cube get multiplied?

Problem 3 Find a formula for the radius of a sphere in terms of its volume. Then use your formula and a calculator to find, to the nearest eighth of an inch, the radius of any sphere that has a volume of 25 cubic feet. Express your answer in terms of feet, inches, and eighths of an inch.

Problem 4 If your answer to the preceding problem is being presented without the statement of the problem at hand, why might it be important to express it in terms of eighths of an inch even if it develops, say, that one gets 2/8 which can be reduced to 1/4?

Problem 5 Is it possible for the intersection of two planes: to be a line? to be a point? to be the empty set?

Problem 6 A pile of dirt in the shape of a circular cone is one yard high, and the diameter of the base is one yard. The dirt is to be spread at a depth of 2 inches to create a new lawn. How may square feet will it cover? First obtain an exact formula, which will involve the number π. Then as a last step use a calculator to obtain an approximation to the nearest square foot.

Problem 7 A square pyramid has a $2'$-by-$2'$ square base and its lateral faces are equilateral triangles. This pyramid is placed on top of a cube having edge length $2'$, with the square base of the pyramid matching the top of the cube. Calculate the volume of the solid thus obtained. Also, calculate the measure of the angle formed by a vertical edge of the cube and a slanting edge of the pyramid.

Problem 8 The latitude and longitude of the Twin Cities are approximately 45° North and 93° West, respectively. Consider travel to the point on the Earth located at 45° North and 87° East. One route is to stay at 45° North the entire trip. Another is over the North Pole on a circular arc centered at the center of Earth. Calculate the distance that would be traveled on each of these two routes, assuming that the surface of the Earth is a sphere of radius 3957 miles.

Problem 9 Rotate a geometrical object 180° about an axis. Then rotate it again through 180°, this time about an axis that is perpendicular to the first axis. Describe the result of these two successive rotations as an appropriate single rotation. If necessary use a physical object as an aid in solving this problem.

Problem 10 Make a model of a regular octahedron by cutting a piece of cardboard, drawing fold-lines on the piece, then folding and taping at appropriate places.

Problem 11 For a regular tetrahedron, draw either front, top, and side views (orientation is your choice) or else a planar graphical representation that accurately portrays the numbers of edges and faces, but not necessarily the areas of the faces.

Problem 12 For this problem refer to the discussion of gravitation near the end of Chapter 3. Assuming that the surface of the earth is a sphere whose equator has length approximately 4.00×10^4 kilometers and that the force between the earth and an object relatively near it is approximately the mass of that object multiplied by 9.81×10^2 centimeters/seconds^2, show that the average density of the earth is approximately 5.5 grams per cubic centimeter. [A theorem due to Issac Newton indicates that for this problem, it is reasonable to assume that all the mass of the earth is at one point—namely at the center of the sphere, so that r in the formula in Chapter 3 is approximately the radius of the earth.]

Chapter 14

Discrete Mathematics

Discrete mathematics, as used here, encompasses a bit of elementary number theory and elementary combinatorics.

Another aspect of discrete mathematics is graph theory. It is my opinion that knowledge of it should not be regarded as an essential part of a solid K-12 mathematics education designed to prepare students to take calculus their first semester in college. However, it is a topic with puzzle-type features that can be fun and motivating for some students, and it has connections with solid geometry that can aid in understanding three-dimensional objects. This last aspect is related to Problem 11 of Chapter 13.

Modular arithmetic is an interesting aspect of number theory, but I do not regard as an integral part of preparation for a college program containing a significant mathematics component. However, it has significant pedagogical value and can help students understand the relations among various aspects of arithmetics and algebra.

No graph theory or modular arithmetic problems are included below.

Problem 1 How many five-digit numerals are there all of whose five digits are different?

Problem 2 In a class of twenty students, there are five students who weigh over 160 pounds and seven students who are at least 5' 11" tall. Among the five students who weigh over 160 pounds there are exactly three who are also at least 5' 11" tall.

(i) How many students are at least 5'11" tall or weigh over 160 pounds?

(ii) Draw a Venn diagram to illustrate the calculation you made for the preceding part.

(iii) How many students weigh no more than 160 pounds and are also less than 5' 11" tall?

Problem 3 Without looking at a book, write down a few rows of Pascal's triangle. Then explain how it is related to combinations and to the Binomial Theorem.

Problem 4 A club consists of 20 men and 14 women. How many ways are there to choose a committee consisting of 4 club members— 2 men and 2 women? At first you may leave the combinations symbol (sometimes called the choose symbol) in your answer. After that, simplify your answer, using a calculator if you like.

Problem 5 Use prime factorizations to calculate the greatest common divisor and least common multiple of 3150 and 5070.

Problem 6 Prove that the square of every odd positive integer leaves a remainder of 1 when divided by 4. Do more: prove that division by 8 also gives a remainder of 1.

Chapter 15

Probability and Statistics

Probability theory is important in many aspects of human endeavor, but often the more frivolous contexts of games of chance and gambling are used for examples. This is done to aid the probabilistic intuition of the student without requiring extensive training in the many various areas where probability theory can be applied. Thus, the game-type problems often used in the learning of probability theory are important for many people, including those who have a religious or common-sense aversion to gambling.

Problem 1 A fair coin is flipped five times in succession. Calculate the probability that:

(i) all five flips give heads;

(ii) the first flip shows heads;

(iii) the fourth flip shows heads;

(iv) no two consecutive flips have the same result.

Problem 2 A deck of 20 labeled cards consists of ten pairs of two cards each; the two cards in a pair have the same label from 1 to 10 as each other, but different from the labels on the other 18 cards. The deck is shuffled randomly . Calculate the probability that:

(i) the top card has the label 3;

(ii) the top two cards both have the label 3;

(iii) the top cards have the labels 3 and 7 in some order;

(iv) the second card from the top has the label 3;

(v) both cards with the label 2 are above both cards having the label 6.

Problem 3 A number is chosen at random according to a normal distribution. Approximate the probability that the number chosen is at least one standard deviation larger than the mean of the distribution. [Recall that the probability that the number is within one standard deviation (either side) of the mean is approximately 0.68.]

Problem 4 Find an approximate equation of the line that best fits, in the sense of least squares, the following four data points:

$$(2.34, 3.12)\,, \quad (3.11, 1.73)\,, \quad (4.97, -1.01)\,, \quad (1.52, 4.33)\,.$$

Of course, only an approximate answer is expected; there is no reason for the calculations to have more accuracy than the three-significant figures in the data. An important aspect of this problem is to organize the solution so that the reader will be able to follow and check the solution using a calculator for nothing except arithmetic.

Chapter 16

Listening to Discussions about Mathematics

In learning mathematics from lectures and discussions it is important to have a wide variety of facts and easy manipulations readily available in one's mind, so as to be able use them for understanding the new concepts and ideas being presented. It is also important to have developed the skill of keeping early aspects of a discussion in mind in order to make connections with later aspects. Two fictitious conversations are given below. Try to follow them without using paper and pencil or calculator. It would be best if you do not read them but have a couple friends read them while you listen with care and intensity. Although both are presented as discussions between two people, a teacher and a student, they could, with slight alterations, be turned into lectures or into conversations among several people.

First Conversation:

Ms. Abernathy: What is the total cost of 63 pens if each pen cost 57 cents?

Jason Corbett: It is the product of 63 and 57, whatever that is.

Ms. Abernathy: Is it that many dollars?

Jason: No, that many cents. The decimal point has to be moved two places to the left to get the numbers of dollars.

Ms. Abernathy: Good. So, the answer in cents is the product of 57 and 63 —that is, 3591 cents or 35 dollars and 91 cents.

Jason: How did you do that multiplication so quickly?

Ms. Abernathy: Start with 60 times 60.

Jason: I can do that: 3600.

Ms. Abernathy: We want to calculate the product of 60 minus 3 and 60 plus 3. This is of the form : the product of x minus y and x plus y. We know that equals x squared minus y squared. What should we choose for x and y in order to do our problem?

Jason: 60 for x and 3 for y. I see: the answer if 3600 minus 9 which equals 3591.

Ms. Abernathy: All of you should now calculate the product of 73 and 87, matching your calculation with the general algebraic identity mentioned earlier.

Second Conversation:

Mr. Garcia: Is the triangle with side lengths 7, 24, and 25 a right triangle?

Carol Jacobson: We only need check whether the square of 25 equals the sum of the squares of 7 and 24. If 'yes' then the triangle is a right triangle; if no, then it is not.

Mr. Garcia: You seem to be saying that the Pythagorean Theorem is an 'if and only if' theorem. Is that right?

Carol: I think so, although the way that the theorem is often stated it only says that the square of the hypotenuse of a right triangle equals the sum of the squares of the other two sides.

Mr. Garcia: You are correct; the converse is also true. Can you prove it?

Carol: My older brother Edward once told me that the converse can be proved by something he called the Law of Cosines. I think if I were to try to prove it I would construct a right triangle with side lengths a, b and c where the square of c equals the sum of the squares of a and b, and then compare with a triangle whose edges have these same lengths and which is assumed to be either acute or obtuse.

Mr. Garcia: That approach will work, but let us go on. Is it true that the square of 25 equals the sum of the squares of 7 and 24?

Carol: I can't calculate the squares of 24 and 25 in my head.

Mr. Garcia: You don't need to. Can you tell me how to go from the square of 24 to the product of 24 and 25?

Carol: You mean without knowing the square of 24?

Mr. Garcia: Yes

Carol: Well, you need to go from 24 24's to 25 24's, so I guess you need to add 24 to the square of 24.

Mr. Garcia: That is correct—And how do you go from 24 times 25 to 25 times 25?

Carol: Oh, I see! Now we need to add 25. Before we added 24. The sum is 49, which is the square of 7. The answer is yes.

Mr. Garcia: Do you see what was important? To get from the square of any integer to the square of the next integer, one adds the sum of the two integers to the first square. Can you express this fact algebraically?

Carol: I can. Add n and the quantity n plus 1 to get 2n plus 1. Then add this to the square of n to obtain n squared plus 2n plus 1 which is the square of n plus 1.

Mr. Garcia: What number played the role of 2n plus 1 in the above calculation?

Carol: 7

Mr. Garcia: Think again.

Carol: Oops; the square of 7.

Mr. Garcia: Which integers have the property that their squares can play the role of 2n+1?

Carol: Those whose squares are odd.

Mr. Garcia: That is ...?

Carol: The odd integers.

Mr. Garcia: Let us try it with another odd integer—say 11

Carol: I don't remember what the square of 11 equals.

Mr. Garcia: What does the square of 10 equal?

Carol: 100 —Oh, I see. Earlier we learned that to get from the square of 10 to the square of 11 we add 10 and 11. So the square of 11 equals 121.

Mr. Garcia: Now: I want to know about the side lengths of a right triangle for which it is known that one of the lengths is 11 and the other two lengths are two consecutive integers.

Carol: The square of 11, which is 121, is playing the role of 2n plus 1, so n must be 60. The side lengths are 11, 60, and 61.

Mr Garcia: Here is a problem for all: Find the side lengths of a right triangle for which it is known that one of the lengths is 37 and the other two lengths are two consecutive integers. Do not use a calculator and do only as much writing as you feel is necessary.

Chapter 17

Pre-Seventh Grade; Later Reinforcement Within Math

Some mathematical skills and facts are learned by the end of sixth-grade mathematics and are reinforced by extensive use in later mathematics courses. It is important that the skill level be sufficiently developed by the end of the sixth grade so that this pre-seventh grade material is an aid, not a hindrance, in further mathematics courses.

Problem 1 Convert the following fractions to repeating or terminating decimals without using a calculator:

(i) 5/7

(ii) 1400/17

(iii) 53/210

Problem 2 Convert the decimal $362.424242\cdots$, where the pattern 42 repeats forever, into a quotient of two integers. Similarly, for the decimal $0.5424242\cdots$, also with the repeating pattern 42.

Problem 3 Use a calculator to partially check your answers to the preceding two problems.

Problem 4 Find the sum of the following two lengths:

$$5 \text{ yards, } 1 \text{ foot, } 8 \text{ inches }; \qquad 4 \text{ yards, } 2 \text{ feet, } 11 \text{ inches }.$$

Problem 5 Convert 50 miles per hour to some number of meters per second, approximating to the nearest 1/100 of a meter. You may use a calculator.

Problem 6 With neither pencil and paper nor calculator do the following arithmetic problems in less than 20 seconds each, giving your answer orally.

(i) $55 + 5 + 27$

(ii) 7×11

(iii) $7\frac{1}{4} - 4\frac{3}{4}$

(iv) $17 \div 5$

Problem 7 With neither pencil and paper nor calculator do the following problem in less than 90 seconds, giving both your answer and reasoning orally. There are 18 holes on a golf course. The par on all but three holes is 4. The par on the fifth and sixteenth holes is 3; that on the ninth hole is 5. What is par for the course? [In case you do not know about the game of golf: Par for the course is the sum of pars for the holes.]

Problem 8 Do the following arithmetic problems without using a calculator, writing your answer as a mixed number whose fractional part is in lowest terms.

(i) $4\frac{7}{11} + 17\frac{3}{4}$

(ii) $\frac{7}{10} \div \frac{21}{55}$

Problem 9 Do the following arithmetic problems without using a calculator, writing your answer as a fraction in lowest terms.

(i) $3\frac{5}{12} - 8\frac{3}{16}$

(ii) $\frac{15}{56} \times \frac{96}{87}$

(iii) $(3/4)/12$

Problem 10 Calculate the following quotient, possibly using a calculator for addition, subtraction, and multiplication of integers (but not for decimal approximations), and express your answer as a fraction, not necessarily in lowest terms:

$$209\frac{452}{1171} \div 32\frac{55}{93}\,.$$

Problem 11 Do the following division problem with a calculator:

$$589444.123 \div 7.2\,.$$

Then do the first couple steps of the division by hand as a partial check on your answer. Also do what is necessary to check that your answer is not off by a factor of 10.

Problem 12 Use some edges of the walls, ceiling, or floor in the room in which you are sitting to explain why it is possible for two lines to never meet even if they are not parallel.

Problem 13 Fill in the blank. If someone says that two lines that do not meet must be parallel, that person must be talking about geometry in ________ dimensions.

Problem 14 What is a pentahedron? Describe an example of one and then make a model of the one you will have described by cutting out a piece of paper and folding it at appropriate places. The model you have created might have another name; if so, state that name.

Problem 15 With neither pencil and paper nor calculator, convert the following fractions to percentages, approximating to the nearest percent.

(i) 2/5

(ii) 2/3

(iii) 3/8 In this case explain why the approximation instructions in this problem might leave one in a quandary.

Problem 16 Make sketches by hand of the following figures.

(i) a parallelogram that is neither a rectangle nor a rhombus

(ii) an isosceles triangle that is not equilateral

(iii) a rectangular parallelepiped that has two square faces but which is not a cube [An alternative to giving 3-dimensional perspective on a 2-dimensional piece of paper is to draw front, top, and side views.]

Chapter 18

Pre-Seventh Grade; Later Reinforcement Outside of Math

Certain skills and facts should be learned by the end of sixth-grade mathematics, and many of them are natural candidates for later reinforcement in courses different from mathematics. It is not critical that a student be familiar with all the facts relevant for the following problems, but familiarity with many facts such as these is an important aspect of good number sense.

I want to emphasize my opinion that when a topic or problem is primarily a topic in a subject other than mathematics then that topic or problem should be treated in a class on the other subject rather than in a mathematics class, even though it might contain some mathematics. For instance, Problem 1 appears here because it concerns order of magnitude, but generally the solar system is a topic for a science class, not a mathematics class. Similarly, substantial population studies belong in social studies classes even though Problem 4, Problem 8, and Problem 9 appear here, the last two for the purpose of indicating that organizational skill in handling data should be developed by the end of sixth grade.

Problem 1 Which of the following is the approximate distance from the earth to the moon in miles: $25,000$; $250,000$; $2,500,000$; $25,000,000$; $250,000,000$?

Problem 2 Which of the following is the approximate distance from the earth to the moon in kilometers: 4×10^9, 4×10^8, 4×10^7, 4×10^6, 4×10^5?

Problem 3 What is the approximate latitude of the Tropic of Capricorn? On approximately what date is the sun directly above a point on the Tropic of Capricorn? That date is the beginning of what season in the northern hemisphere?

Problem 4 About what percentage of the population of the world is contained in India and China combined: 0.7%, 2%, 10%, 35%, 70%?

Problem 5 Who of the following was president of the United States four score and seven years ago: George Washington, Abraham Lincoln, Woodrow Wilson, Franklin Roosevelt, Lyndon Johnson?

Problem 6 What is the approximate area of Minnesota in square miles: $9,000$; $90,000$; $900,000$; $9,000,000$; $90,000,000$?

Problem 7 On what day of the week will July 31 fall in the year 2012?

Problem 8 There are four urban areas—Pretend, Imagine, Envision, and Dream—in the country Fictionalplace, the total population of which is 10,340,002. The populations of the urban areas are:

$$\begin{array}{rr} \text{Pretend} & 2,435,778\,; \\ \text{Imagine} & 722,893\,; \\ \text{Envision} & 3,522,801\,; \\ \text{Dream} & 2,006,090\,. \end{array}$$

Draw a pie chart showing the percentages of populations in each of the four urban areas as well as in rural (the remainder of) Fictionalplace.

Problem 9 Make a histogram for the populations of the counties of Minnesota. Arrange for enough equally-spaced categories so that among them there are 10 to 15 containing at least one county each. Then make an approximate check of your work by using the mid-point population of each category to estimate the total population of Minnesota. Also make a list of the counties and their populations, ordered by population, and then find a median of these county populations.

Chapter 19

My Background

From 1951-1955 I attended Hopkins High School when Hopkins was still a semi-rural area west of Minneapolis. I graduated from the University of Minnesota in 1959. In 1963 I received my PhD in mathematics at MIT in Cambridge, Massachusetts, and then joined the University of Minnesota mathematics faculty.

My main research area is probability and I have also done some research in combinatorics (sometimes called discrete mathematics), game theory, and one area within statistics.

At the University I have taught probability at all levels, calculus at all levels, college algebra, the mathematics courses taken by prospective elementary school teachers, a liberal-type mathematics course for those students whose majors have no specific mathematics requirement, and, at the junior-senior level: combinatorics, graph theory, geometry, and abstract algebra.

In special summer programs organized by Prof. Harvey Keynes, I have taught in-service secondary mathematics teachers as well as advanced high school and middle school students.

I have used and continue to use a variety of teaching styles.

I am coauthor, with Prof. Lawrence Gray, of a graduate-level textbook in probability. At an earlier time, Prof. Donald Berry, who at that time was in the Statistics Department at the University of Minnesota, and I wrote a monograph on sequential decision theory. My current writing project is a junior-senior college-level textbook on geometry; a preliminary version has been used by others and myself as a textbook for courses taken primarily by prospective secondary mathematics teachers.