Some Answers and Solutions for the Problems in Desirable Outcomes from a Full K-12 Mathematics Program

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Contents

Chapter 1

Introduction

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This document is a response to some requests for answers to the problems I have previously placed on my web site under the link 'Desirable Outcomes from a Full K-12 Mathematics Program'.

In giving solutions, I want to emphasize that I do not view the individual problems in the list as special. Rather, the collection indicates scope. Memorizing the solutions to the particular problems in the list without developing the skill to do somewhat similar problems of the same level of difficulty would be misplaced effort. It is for this reason that I have been somewhat reluctant to write answers and solutions. Interspersed with the solutions are some of my opinions. They and any opinions implicit in the solutions are, of course, my own and do not reflect an official position of the University of Minnesota.

I want to thank my colleague Professor John Baxter, here in the Mathematics Department at the University of Minnesota, for helping me check some of my solutions. Nevertheless, it would not surprise me if I have made an error or two in a such a long list of answers and solutions; please let me know of any that you find.

A major stumbling block to learning mathematics at any level can be a lack of facility with prerequisite material. In the following three paragraphs I use specific problems to illustrate this point.

Part (iii) of Problem 4 of Chapter 2 is an easy problem (except for the requirement that attention be given to the values of u for which the calculation is valid). However, this problem can be somewhat difficult for a student who has not memorized the Laws of Exponents and the formula for $(a+b)^2$. That he or she might know where to look these up does not remove this element of difficulty, because looking them up takes attention away from the problem-solving process—using the formula for $(a+b)^2$ in combination with one Law of Exponents.

The inequality in Part (ii) of Problem 14 of Chapter 3 makes it a problem of medium difficulty. It becomes an extremely difficult problem for the student who has not previously become comfortable and accurate in removing parentheses.

The hard Problem 11 of Chapter 12 can become a frustration for a person who has not previously become very familiar with the sine and absolute value functions.

Similarly, when studying calculus (and other subjects such as physics which make extensive use of mathematics) the level of difficulty for a student increases significantly in cases where the relevant prerequisite knowledge and skill is skimpy.

Some problems that I have included are not critical for learning calculus, but represent important preparation for studying other subjects. For instance, the hard Problem 11 of Chapter 13, which involves geometrical visualization, can be relevant for disciplines emphasizing design of various physical structures. Problem 4 of Chapter 15 is another example. It is a standard problem for those who have studied least-square approximation and essentially impossible for those who haven't. This topic is not a prerequisite topic for calculus, but it can be useful for those following a path that involves some calculator or computer programs that approximate data with formulas.

When writing solutions I have been rather complete. Thus I treat a variety of subtle issues which might be viewed as beyond the core of some problems. Although I hope that students who have had a full K-12 mathematics education are aware of such issues, I would not generally expect them to identify every relevant occurrence. For instance, the manipulative aspects of Part (ii) of Problem 16 in Chapter 2 are themselves quite difficult and it might be too much to expect most students to be sensitive throughout the calculation to the possibility that both numerator and denominator might have been multiplied by 0 and that elsewhere in the calculation a common factor of 0 might have been removed from both numerator and denominator.

The chapter numbers, chapter titles, and problem numbers here match the corresponding items in 'Desirable Outcomes from a Full K-12 Mathematics Program'. All problem numbers are listed even if no answer or solution is being supplied.

Chapter 2

Algebraic Simplifications

Problem 1 Answers: 0; -96; 5 (not ± 5); 7; 0; 16/3 or $5\frac{1}{3}$

Problem 2 Answers: $-3x^2 + 28x - 6$; a for $a \neq 0$

Problem 3 Values: -221, 51, 2, -2

Problem 4 Solutions:

Part (i):

$$\frac{u-\sqrt{13}}{u^2-13} = \frac{u-\sqrt{13}}{(u-\sqrt{13})(u+\sqrt{13})} = \frac{1}{u+\sqrt{13}}\,,\ u \neq \sqrt{13}$$

[Comment: It is important to mention $u \neq \sqrt{13}$ since the final formula is meaningful for $u = \sqrt{13}$ even though the given formula is not. Of course, $u \neq -\sqrt{13}$ could also be mentioned, but this condition is implicit because the final formula is meaningless for $u = -\sqrt{13}$.]

Part (ii)

$$\frac{2u^2-7u+5}{4u^2-8u-5} = \frac{(2u-5)(u-1)}{(2u-5)(2u+1)} = \frac{u+1}{2u+1}\,,\ u \neq 5/2$$

Part (iii)

$$[3u^{1/2} + 5u^{-1/2}]^2 = 9u + 30 + 25u^{-1}\,,\ u > 0$$

Problem 5 Answers: $\sqrt{13}$; 5/2; negative numbers.

Problem 6 Answer:

$$\tfrac{5}{2}x^2 + 4x - \tfrac{7}{4} + \frac{-\frac{103}{2}x + \frac{101}{4}}{2x^2 - 6x + 11}$$

Here is an alternative form in case one wants no fractions within the numerator:

$$\tfrac{5}{2}x^2 + 4x - \tfrac{7}{4} + \frac{-206x + 101}{8x^2 - 24x + 44}$$

Problem 7 Answer:
$$\frac{9x^5 - 6x^3 + 9x^2 - 8x + 4}{3x^2 + 2}$$

Problem 8 Values: 136/31; −92/7

Problem 9 Answers: 4; $\log_b 70$; −3

Problem 10 Answers: 403.43; 0.69315; 0.52110; −1.2528

Problem 11 Solution: The relevant term in the expansion is
$$\frac{9!}{7!\,2!}\,a^7(3b)^2 = 36 \cdot 9a^7b^2 = 324a^7b^2\,.$$
The answer is 324.

Problem 12 Answer: $-5 < x < 9$

Problem 13 Answers: $\{3,5\}$; the empty set $\emptyset$; $\{2,3,5,8,32\}$; $\{2,3,5,8,32\}$

Problem 14 Answer: $(-\infty,-8] \cup [-2,\infty)$

Problem 15 Answer: $(1,2)$

Problem 16 Solutions:

Part (i):
$$\begin{aligned}\frac{3x+\sqrt{9x^2-5}}{3x-\sqrt{9x^2-5}} &= \frac{\left(3x+\sqrt{9x^2-5}\right)^2}{\left(3x-\sqrt{9x^2-5}\right)\left(3x+\sqrt{9x^2-5}\right)}\\ &= \frac{9x^2+6x\sqrt{9x^2-5}+(9x^2-5)}{9x^2-(9x^2-5)}\\ &= \frac{18x^2-5+6x\sqrt{9x^2-5}}{5}\,.\end{aligned}$$

Part (ii):
$$\begin{aligned}\frac{(x^3-y^3)\,(y-x)}{2xy-\sqrt{2x^4+2y^4}} &= \frac{\left(x^3-y^3\right)\left(y-x\right)\left(2xy+\sqrt{2x^4+2y^4}\right)}{\left(2xy-\sqrt{2x^4+2y^4}\right)\left(2xy+\sqrt{2x^4+2y^4}\right)}\\ &= \frac{\left(x^3-y^3\right)\left(y-x\right)\left(2xy+\sqrt{2x^4+2y^4}\right)}{4x^2y^2-(2x^4+2y^4)}\\ &= \frac{\left(x^3-y^3\right)\left(y-x\right)\left(2xy+\sqrt{2x^4+2y^4}\right)}{-2(x^2-y^2)^2}\\ &= \frac{\left(x-y\right)\left(x^2+xy+y^2\right)\left(y-x\right)\left(2xy+\sqrt{2x^4+2y^4}\right)}{-2(x-y)^2(x+y)^2}\\ &= \frac{\left(x^2+xy+y^2\right)\left(2xy+\sqrt{2x^4+2y^4}\right)}{2(x+y)^2}\,.\end{aligned}$$

For the solution to be complete some remarks must be make concerning the case $|y| = |x| \neq 0$. If $y = x \neq 0$, the given formula is meaningless even though the formula obtained by the above calculations is meaningful. If $y = -x \neq 0$, the given formula is meaningful (it has the value x^2) even though the formula obtained via the above calculations is meaningless. [Comments: Nothing need be said about the case $y = x = 0$ since neither the given formula nor the final formula is meaningful in that case. In the case $y = x$, the cancellation of $y - x$ near the end of the calculation created meaningfulness from meaninglessness. In the case $y = -x$, the early multiplication of both numerator and denominator by a quantity which is 0 in this case created meaninglessness from meaningfulness. No further simplification of the final expression obtained in the above calculation seems to result from multiplying out the numerator, but it is also ok to do that multiplication. Finally, notice that it was wise to do little with the numerator until we were able to observe which factors would occur in the denominator.]

Problem 17 Solution: Multiply both numerator and denominator by $5 + 4i$ to obtain

$$\frac{15 + 22i + 8i^2}{25 - 16i^2} = \tfrac{7}{41} + \tfrac{22}{41} i\,.$$

Problem 18 Algebraic Solution: We are looking for real numbers x and y such that

$$(x + yi)^2 = 3 + 4i\,,$$

that is,

$$\begin{aligned} x^2 - y^2 &= 3 \\ 2xy &= 4\,. \end{aligned}$$

Since $x = 0$ will not work in the second equation, we do not lose any solutions by dividing both sides of the second equation by $2x$ to obtain $y = 2/x$. Substituting this expression for y in the first equation we get

$$x^2 - \frac{4}{x^2} = 3\,.$$

Multiply both sides by x^2 and rearrange to obtain:

$$x^4 - 3x^2 - 4 = 0\,.$$

This quadratic equation in x^2 can be solved by factoring:

$$(x^2 - 4)(x^2 + 1) = 0\,.$$

The only positive solution for x^2 is 4. Thus $x = \pm 2$ and $y = 2/(\pm 2) = \pm 1$, with the understanding that x and y are either both positive or both negative. The two square roots are

$$2 + i \qquad \text{and} \qquad -2 - i\,.$$

Trigonometric Solution (as we step out of the spirit of this chapter): A consequence of $3^2 + 4^2 = 5^2$ is that the given complex number can be written as

$$5\cos\theta + 5i\sin\theta$$

where $0 < \theta < \pi/2$ radians and $\cos\theta = 3/5$. It follows from a standard fact that one of the desired square roots equals

$$5^{1/2}\cos(\theta/2) + 5^{1/2} i \sin(\theta/2)\,.$$

Using the half-angle formulas we get

$$\cos(\theta/2) = \sqrt{\frac{1+\cos\theta}{2}} = \sqrt{\frac{4}{5}} = \frac{2}{5^{1/2}}$$

and

$$\sin(\theta/2) = \sqrt{\frac{1-\cos\theta}{2}} = \sqrt{\frac{1}{5}} = \frac{1}{5^{1/2}}\,.$$

When these expressions are inserted into the formula obtained above for the square root that formula simplifies to $2+i$. The other square root is its negative—namely, $-2-i$. [Comment: It was clear that $0 < \theta/2 < \pi/2$, so it was appropriate to use the positive square roots when using the half-angle formulas.]

Problem 19 Answers: 125; 1/125; 49

Problem 20 Answers: 12; twice the cube of x; the product of 3, the square root of 3, and the absolute value of u; the tenth power of v; 3/4; the sixth power of b less the sixth power of a

Problem 21 Answers: $10\sqrt[3]{2}$; x^5 for $x \neq 0$

Problem 22 Answers: Part (i) 11 and 12 because $11^2 = 121 < 135$ and $12^2 = 144 > 135$. Part (ii) It lies between 110 and 120 since multiplying a number by 100 multiplies its square root by $\sqrt{100} = 10$.

Problem 23 Answer: 28

Problem 24 Solution: 2.6833. The best five-significant-digit approximation of 2.6833^2 is 7.2001 which differs from 7.2000 in the fifth digit—so the answer to the query is 'no'.

Problem 25 Solutions: $\sqrt{72000000} = \sqrt{72} \times 10^3$ which is approximately equal to 8.4853×10^3. The best five-significant-digit approximation of $(8.4853 \times 10^3)^2$ is 7.2000×10^7 which agrees with 72000000 in the first five digits. In so far as the second expression is concerned, it can be written as $\sqrt{7.2 \times 10^{28}}$. From the previous problem (not the preceding part of this problem) we obtain the best five-significant-digit approximation of $\sqrt{72 \times 10^{27}}$ —namely, 2.6833×10^{14} whose five-significant-digit square differs from 72×10^{27} in its fifth position.

Problem 26 Solutions: A calculator gives $7.2^{3/7} = 2.3304$, correct to five significant digits. It also gives $2.3304^{7/3} = 7.2001$, correct to five significant digits, which does not agree with the given 7.2 in the fifth position. Since $(10^7)^{3/7} = 10^3$ and $72000000 = 7.2 \times 10^7$ we conclude that

$$72000000^{3/7} = 2.3304 \times 10^3 = 2330.4\,,$$

accurate to five significant digits. Similarly since $\left(10^{28}\right)^{3/7} = 10^{12}$ and $72 \times 10^{27} = 7.2 \times 10^{28}$, we conclude that

$$\left(72 \times 10^{27}\right)^{3/7} = 2.3304 \times 10^{12}\,,$$

accurate to five significant digits. The $7/3^{\text{rd}}$ power, accurate to five significant digits, of these answers will differ from the numbers in the problem by 1 in the fifth position.

Problem 27 Answer: 4.278543

Problem 28 Answer: -0.95015

Problem 29 Solution: 17 feet, 6.9 inches equals 210.9 inches. As obtained from a table, one inch equals approximately 2.540005 centimeters, accurate to seven significant digits. It follows that 210.9 inches is approximately 535.6871 centimeters which can be written as 5 meters, 3 decimeters, 5 centimeters, 7 millimeters to the nearest millimeter. [Even though the answer contains only four significant digits it would not have been good practice to have only kept four significant digits throughout the calculation. Successive rounding-off as the calculation proceeds could then cause the ultimate fourth digit to be in error.]

Chapter 3

Algebraic Equations

Problem 1 Answers: $-47\frac{1}{4}$; 1

Problem 2 Answers and Solutions:

Part (i):

$$v = \frac{-8 \pm \sqrt{79}}{3}$$

Part (ii): Using the value of v corresponding to the '+' sign, the left side of the given equation is

$$\begin{aligned} &3\Big(\frac{-8+\sqrt{79}}{3}\Big)^2 + 14\Big(\frac{-8+\sqrt{79}}{3}\Big) \\ &= \Big(\frac{64 - 16\sqrt{79} + 79}{3}\Big) + \Big(\frac{-112 + 14\sqrt{79}}{3}\Big) \\ &= \frac{143 - 112 - 2\sqrt{79}}{3} = \frac{31 - 2\sqrt{79}}{3}\,; \end{aligned}$$

and the right side of the given equation becomes

$$\begin{aligned} &-2\Big(\frac{-8+\sqrt{79}}{3}\Big) + 5 \\ &= \Big(\frac{16 - 2\sqrt{79}}{3}\Big) + \frac{15}{3} = \frac{31 - 2\sqrt{79}}{3}\,. \end{aligned}$$

The two calculations have given the same result, as desired.

The calculation for the other solution is similar.

Part (iii): 0.29606, −5.62940

Part (iv):

$$3(0.29606)^2 + 14(0.29606) = 4.40779\,, \quad -2(0.29606) + 5 = 4.40788$$

and

$$3(-5.62940)^2 + 14(-5.62940) = 16.25883\,, \quad -2(-5.62940) + 5 = 16.25880$$

—all four calculations correct to five decimal places. [Comment: The solutions that are correct to five decimal places do not make the two sides of the given equation equal in the fifth decimal place and for the answer 0.29606 rounding-off to the fourth decimal place gives the numbers 4.4078 and 4.4079 which do not even agree in the fourth decimal place. This disagreement does not mean that the answer 0.29606 is in error.]

Problem 3 Answers: $|k| > 10\sqrt{3}$; $k = \pm 10\sqrt{3}$; $-\sqrt{3}$ if $k = 10\sqrt{3}$ and $\sqrt{3}$ if $k = -10\sqrt{3}$

Problem 4 Answers: $(10 + 2a)/(5 - a)$ if $a \neq 5$ and no solutions if $a = 5$; $1/1000$; $(\log 9)/(\log 2)$ —any base will do, but the same base must be used for both logarithms; 5; no solutions

Problem 5 Answer: 3.16993 according to my calculator. [Comment: My calculator actually showed 3.169925001. If this is correct, then I have rounded-off correctly to get 3.16993. But has my calculator made a rounding-off error itself? It is conceivable that it used ten-significant-digit approximations of $\log 9$ and $\log 2$ and then performed the division. If so, the final '1' in 3.169925001 could be in error, in which case it might be that 3.16992 would be the best six-significant-digit approximation. Alternatively, my calculator might use more significant digits internally than it actually shows. If, say, it used twelve-significant-digit approximations for $\log 9$ and $\log 2$, did the division, and finally rounded-off to ten significant digits, the last '1' would definitely be correct and 3.16993 would, without doubt, be the best five-decimal-place approximation.]

Problem 6 Prelude: Since the problem asks for a solution set I will use set notation even in the two cases where there is exactly one solution. Answers: $\{(-\frac{15}{16}, -\frac{11}{16})\}$; $\{(-1, -2), (-\frac{2}{29}, \frac{50}{29})\}$; $\{(-\frac{\log 11}{\log 5}, -\frac{15}{11})\}$ —any base for the logarithms is ok but the same base must be used for both logarithms.

Problem 7 Prelude: Since the next problem concerns itself with certain subtle issues, those issues will not be discussed in the answers here. Answers:

Part (i): $x = z4^{-1/y}$, $y = \frac{\log 4}{\log(z/x)}$, $z = x4^{1/y}$

Part (ii): $x = \frac{z \pm \sqrt{z^2 - 60y}}{6}$, $y = \frac{zx - 3x^2}{5}$, $z = \frac{3x^2 + 5y}{x}$

Part (iii): $x = 1/(yz)$, $y = 1/(zx)$, $z = 1/(xy)$

Problem 8 Solution for Part (i): Response to (a); z/x must be positive and y can be any real number. However, if $y > 0$, then $z = 0$ is permitted and if y is an integer, then z/x can be negative. Response to (b): $y \neq 0$ for the formulas for x and for z. The formula for y requires z/x to be positive and different from 1.

Problem 9 Answers: Part (i): $\frac{7 \pm i\sqrt{11}}{6}$; Part (ii): 1, $\frac{3 \pm \sqrt{57}}{4}$; Part (iii)

$$\pm\sqrt{\frac{-3 + \sqrt{149}}{2}}, \quad \pm i\sqrt{\frac{3 + \sqrt{149}}{2}}$$

Problem 10 Answers: $\{(2,-2,2)\}$, $\{(\frac{8}{3}-2t.\frac{1}{3}+t,t)\colon t \text{ arbitrary}\}$

Problem 11 Remark: Real numbers are special cases of complex numbers. Answer: Every polynomial equation in one variable with complex coefficients has at least one complex solution provided that the polynomial is not a constant polynomial. [Comment: An answer which assumes that the polynomial has real coefficients should be treated as correct; I do not view polynomials with complex coefficients as constituting a standard K-12 topic.]

Problem 12 Answers: $r=\sqrt{Gm_1\,m_2/F}$; $m_1 = Fr^2/(Gm_2)$

Problem 13 Answer: inversely, square, distance

Problem 14 Answers: $[-\frac{17}{5},\infty)$; $(-7,4)$; $(-\infty,-2]\cup\{5\}$; the empty set $\emptyset$

Problem 15

Problem 16

Chapter 4

Algebraic Proofs

Problem 1 Solution: The given inequality is equivalent to

$$0 \le x^2 - 2xy + y^2 \,,$$

which itself is equivalent to

$$0 \le (x-y)^2 \,,$$

a true statement for all real numbers x and y, because the square of any real number is nonnegative. [Comment: In view of the fact that the above proof consists solely in showing various statements to be equivalent, it is legitimate that we started with the statement that we were to prove. Generally, one has to be very careful in doing manipulations and arguments based on the statement one is trying to prove.]

Problem 2 Solution: The given information is equivalent to

$$\frac{c}{d} - \frac{a}{b} > 0 \,,$$

which is equivalent to

$$\frac{bc - ad}{bd} > 0 \,.$$

Since b and d are positive the last inequality is equivalent to

$$bc - ad > 0 \,.$$

This is what we are given. Similar calculations show that what we want to prove is equivalent to

$$b(a+c) - a(b+d) > 0 \,.$$

But this desired inequality reduces to $bc - ad > 0$, which, as mentioned earlier, is equivalent to what we are given.

Problem 3 Answers: no; yes; no

Problem 4 Solutions: Let a denote the number of correct answers on the first part, b the number of problems on the first part, c the number of correct answers on the second part, and d the number of problems on the second part. Then $100a/b$, $100c/d$, and $100(a+c)/(b+d)$ are the percentages correct on the first part, the second part, and the entire test, respectively. The assertion in Part (ii) now follows immediately from Problem 2 by multiplying all fractions in that problem by 100. Also, a proof for Part (i) can be obtained by replacing each '$>$' in the solution of Problem 2 by '$=$'.

Chapter 5

Exponents in Applications

Problem 1 Answer: \$1191.02, which is larger than \$1180.00, as it should be because \$1180.00 would be the value of the investment after three years had there been no compounding.

Problem 2 Solution: We first solve the equation $(1.08)^t = 3$. We get

$$t = \frac{\log_{10} 3}{\log_{10} 1.08},$$

which is approximately 14.3. The answer is 15 years. [Comments: The rounding-off in this problem has to be rounding-up, not rounding-to-the-closest; at the end of the fourteenth year the investment will not have yet tripled in value and the same is true at times during the fifteenth year since interest is only awarded at the completion of the fifteenth year. The logarithms in the calculation could be with respect to any base, but the same base has to be used in numerator and denominator.]

Problem 3 Solution: $6 \times 4 + 1 = 25$ interest periods are involved. The rate per period is one-fourth of the annual rate—namely, 1.25%. The answer is

$$\$3000(1.0125)^{25} = \$4092.58$$

to the nearest cent.

Problem 4 Answer:

$$\frac{\log_e B - \log_e A}{\alpha - \beta},$$

which could also be written as

$$\frac{\ln B - \ln A}{\alpha - \beta},$$

where ln denotes the natural logarithm

Problem 5 Answer: $(\ln 2)/t_0$, where, as in the preceding answer, ln denotes the natural logarithm [Comment: $\ln 2$ is approximately 0.693.]

Chapter 6

Plane Geometry

This might be a good place for a reminder. A student who has learned the relevant material in a course might still need to brush up on that material before tackling some of the more difficult problems related to that course.

Different notations are used in various books. In the solutions here PQ will denote the line through distinct points P and Q. The line segment with endpoints P and Q will be denoted by $\overline{PQ}$.

Problem 1 Answer: The square of the length of the hypotenuse of a right triangle equals the sum of the squares of the lengths of the other two sides.

Problem 2 Solution: Since $20^2 + 21^2 = 400 + 441 = 841 = 29^2$ we suspect that a triangle with side lengths 20, 21, and 29 is a right triangle. This is not a consequence of the Pythagorean Theorem as written for the preceding problem. However, it is a consequence of the following statement which is also a theorem. If the square of the length of one side of a triangle equals the sum of the squares of the lengths of the other two sides, then the triangle is a right triangle and the longest side is its hypotenuse. Some would include this last statement in the Pythagorean Theorem. Then the statement of the Pythagorean Theorem would be longer than the answer given above for the preceding problem, and the solution of this problem would be shorter.

Problem 3 Solution: In inches the length of the hypotenuse is

$$\sqrt{876^2 + 1133^2} = 1432$$

to the nearest inch; this equals 119 feet, 4 inches. The measure of the angle opposite the hypotenuse is, of course, 90°. The tangent of the angle opposite the side of length 1133 inches equals

$$\frac{1133}{876},$$

which is approximately 1.29338. The measure of this angle is approximately 52° 20 minutes. Since the sum of the angular measures equals exactly 180°, the measure of the remaining angle equals 37° 40 minutes. [Comment: When using the calculator there is no need to write down the tangent of the angle.]

Problem 4 Answers: $(n-2)180^\circ$. This answer does not depend on the polygon being regular. [Comment: However, the answer does depend on the polygon being convex. A way of making the formula valid for non-convex polygons is to define 'interior angle' whose measure might exceed 180° —then the formula is correct for the sum of the measures of the interior angles of an arbitrary n-sided polygon.]

Problem 5 Prelude: Two proofs will be given. The first matches points with ordered pairs of numbers and relies on algebraic manipulations. The second is a standard Euclidean-type proof in which coordinates do not appear. It is not expected that students would be comfortable with both methods. It also might be that a student has used one type of proof for some types of geometrical statements and the other method for other types. I also mention, for those who have studied vector algebra, that a proof using vectors is possible, but will not be included here.

Algebraic proof: Let $\triangle ABC$ be an arbitrary triangle. Introduce coordinate axes in such a way that $C=(0,0)$ and $A=(r,0)$ for some positive r. Denote the coordinates of B by (s,t). Notice that $t\neq 0$, a consequence of the fact that A, B, and C are not collinear.

The slopes of AB, BC, and CA are

$$\frac{t-0}{s-r}=\frac{t}{s-r}\,, \qquad \frac{0-t}{0-s}=\frac{t}{s}\,, \qquad \text{and } \frac{0-0}{a-0}=0\,,$$

respectively. Therefore, the slopes of the perpendicular bisectors of $\overline{AB}$ and $\overline{BC}$ are

$$\frac{r-s}{t} \quad \text{and} \quad \frac{-s}{t}\,,$$

respectively—and the slope of the perpendicular bisector of $\overline{CA}$ is not defined.

The midpoints of $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$ are

$$(\tfrac{r+s}{2},\tfrac{0+t}{2})=(\tfrac{r+s}{2},\tfrac{t}{2})\,, \quad (\tfrac{s+0}{2},\tfrac{t+0}{2})=(\tfrac{s}{2},\tfrac{t}{2})\,, \quad \text{and } ((\tfrac{0+r}{2},\tfrac{0+0}{2})=(\tfrac{r}{2},0)\,,$$

respectively.

Using the above information we can write the equations of the perpendicular bisectors:

$$\begin{aligned} y-\tfrac{t}{2} &= \tfrac{r-s}{t}(x-\tfrac{r+s}{2})\,,\\ y-\tfrac{t}{2} &= \tfrac{-s}{t}(x-\tfrac{s}{2})\,,\\ x &= \tfrac{r}{2}\,. \end{aligned}$$

Insertion of $\frac{r}{2}$ for x in the first two of these equations yields the same value for $y-\frac{t}{2}$ —namely,

$$y-\tfrac{t}{2}=-\tfrac{(r-s)(s)}{2}\,.$$

Therefore, the three perpendicular bisectors are concurrent at the point

$$(\tfrac{r}{2},\tfrac{t-(r-s)s}{2})\,.$$

Euclidean-type proof: Denote an arbitrary triangle by $\triangle ABC$ and denote the perpendicular bisectors of $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$ by l, m, and n, respectively. By a basic theorem, l consists of those points that are equidistant from B and C, and

m consists of those points that are equidistant from C and A. Also, l and m do meet somewhere (not necessarily in the interior of the triangle) because they are perpendicular to two non-parallel lines. Denote the point where they meet by J —it is equidistant from B and C and also equidistant from C and A. By the transitive property of equality, J is equidistant from B and A, Thus it lies on n since n consists of those points that are equidistant from B and A. Therefore, the three perpendicular bisectors meet at J. This completes the proof.

Presumably, when one is being asked to prove the theorem that the three perpendicular bisectors of the sides of a triangle meet at a point, one is to assume that the basic properties of perpendicular bisectors of segments are known and have previously been proved. Nevertheless, I will state and prove two such basic properties, again using a classical Euclidean approach rather than coordinates.

Here is one of the basic properties. The points on the perpendicular bisector l of a segment $\overline{BC}$ are equidistant from B and C. Notice that l is not mentioned by the name l in the proof that follows; rather it goes by the name PD with P being an arbitrary point on l.

Given: B, C, and D are collinear
$\overline{BD} \cong \overline{CD}$
$P \neq D$ and $PD \perp BC$
To prove: $\overline{BP} \cong \overline{CP}$

Statements	Reasons
$\overline{BD} \cong \overline{CD}$	given
$\angle BDP \cong \angle CDP$	$BC \perp PD$ and congruence of any two right angles
$\overline{DP} \cong \overline{DP}$	obvious given that $P \neq D$
$\triangle BDP \cong \triangle CDP$	side-angle-side
$\overline{BP} \cong \overline{CP}$	corresponding parts of congruent triangles

[Comment: The two-column proof I have just given is incomplete. Nowhere is it mentioned that D itself is both on the perpendicular bisector and equidistant from B and C. Similarly, in the upcoming proof it is not mentioned that D is the only point on BC that is equidistant from B and C. Little extra comments such as these, which are often needed to make proofs complete, are not easily incorporated into two-column proofs without masking the main issues. On the other hand, two-column proofs have some advantages over paragraph-type proofs; they tend to enforce self-discipline on the proof writer, a self-discipline that all good proof writers must have.]

Here is the second basic property to be proved, which is the converse to the first statement. Every point that is equidistant from two points B and C lies on the perpendicular bisector of BC.

Given: 1. B, C, and D are collinear
2. $\overline{BD} \cong \overline{CD}$
3. Q is not on BC
4. $\overline{BQ} \cong \overline{CQ}$

To prove: $QD \perp BC$

Statements	Reasons
5. $\overline{BD} \cong \overline{CD}$	(2)
6. $\overline{DQ} \cong \overline{DQ}$	obvious given (3)
7. $\overline{BQ} \cong \overline{CQ}$	(4)
8. $\triangle BDQ \cong \triangle CDQ$	$(5), (6), (7)$ and SSS
9. $\angle BDQ \cong \angle CDP$	(8)
10. $QD \perp BC$	(9) and (1)

[Comment: For the three Euclidean proofs I have used a paragraph style once and a two-column style twice, including some style differences between the two two-column proofs. It is worth emphasizing that proofs having very different appearances can all be correct proofs of a theorem. (And there can also be two somewhat similar appearing 'proofs' of the same fact with one of the proofs being correct and the other incorrect.)]

Problem 6 Answer: Obtuse triangles—that is, those triangles having one obtuse angle. [Comment: What is given here is only an answer. A full solution should contain a proof that the circumcenter of an obtuse triangle is exterior to the triangle and also a proof that the circumcenter of a right triangle or an acute triangle is not exterior to the triangle.]

Problem 7 Answer: side-side-angle

Problem 8

Problem 9 Solution: Label the vertices of the rhombus by A, B, C, and D in such a way that the diagonals are $\overline{AC}$ and $\overline{BD}$. Let E denote the point of intersection of $\overline{AC}$ and $\overline{B}D$. By basic knowledge of parallelograms we have that $\overline{AE} \cong \overline{CE}$. We also note that $\overline{BE} \cong \overline{BE}$ since any segment is congruent to itself. Since the figure of concern is a rhombus, $\overline{AB} \cong \overline{CB}$. By the SSS-criterion for two triangles to be congruent, $\triangle ABE \cong CBE$. Therefore, $\angle AEB \cong \angle CEB$. Since A, C, and E are collinear it follows that the two congruent angles $\angle AEB$ and $\angle CEB$ are both right angles.

Problem 10 Solution: Step 1: Set the compass equal to the length of the given line segment.

Step 2: Use the compass to draw two circles centered at the two endpoints of the given line segment.

Step 3: The circles drawn in Step 2 will have two points of intersection. Use the straightedge to draw the line through those two points—this line is the desired perpendicular bisector.

[Comment: In setting the compass it is not necessary that Step 1 be used. However, it is important that it be set at a length large enough for the two circles drawn in Step 2 to have two points of intersection.]

Problem 11 Answer: $(0, -3)$

Problem 12 I made a mistake by including this problem. It requires too much ingenuity. It has been removed from the problem list on my web site.

Problem 13 Let m be a line and P a point not on m. Then there exists exactly one line through P that does not meet m. [Here is an alternative second sentence: Then there exists no more than one line through P that does not meet m. Reason that alternative is ok: It can be proved from the other axioms that there is at least one such line.]

Problem 14 Statement: The measure of $\angle ACB$ is twice that of $\angle ADB$.

Proof: The measure of $\angle ADB$ is the sum of the measures of $\angle ADC$ and $\angle BDC$. Because $\triangle ADC$ and $\triangle BDC$ are isosceles, $\angle ADC \cong \angle DAC$ and $\angle BDC \cong \angle DBC$. Thus the measure of $\angle ADB$ also equals the sum of the measures of $\angle DAC$ and $\angle DBC$.

From the preceding paragraph we deduce that twice the measure of $\angle ADB$ equals the sum of the measures of the following four angles:

$$\angle ADC\,, \quad \angle DAC\,, \quad \angle BDC\,, \quad \angle DBC\,.$$

The sum of the measures of the first and second of these angles also equals 180° less the measure of $\angle ACD$ and the sum of the measures of the third and fourth of them equals 180° less the measure of $\angle BCD$. Thus the sum of all four of the measures of these angles not only equals twice the measure of $\angle ADB$ but also equals 360° less the sum of the measures of $\angle ACD$ and $\angle BCD$. Because 360° is the sum of all the angular measures at a point, we finish the proof by noting that 360° less the sum of the measures of $\angle ACD$ and $\angle BCD$ equals the measure of $\angle ACB$.

[Comments: The above argument would be less cumbersome had I used some symbol to represent measure of an angle. Concerning style, some might have preferred that I had explicitly said at the beginning of the proof: 'Connect C and D with a line segment.' This line segment appears implicitly in the proof at the first appearance of $\angle ADC$.

Chapter 7

Algebra Used for Geometry

Problem 1 Answers: $5x+3y=19$, which can be expressed in a variety of equally simple ways; x-axis at 19/5 and y-axis at 19/3

Problem 2 Answer: $x=-3$

Problem 3 Answer: $-3/5$

Problem 4 Answers and Solution: $(-\frac{3}{5},0)$ and $(0,\frac{1}{4})$; $\frac{3}{40}$

Part (iii): Let d denote the distance from $(0,0)$ to the line $5x-12y=-3$, and c the length of the hypotenuse of the triangle. From Part (ii) we obtain

$$\tfrac{1}{2}cd=\tfrac{3}{40}$$

from which we conclude that

$$d=3/(20c)\,.$$

We use the answer to Part (i) to calculate c:

$$c=\sqrt{(3/5)^2+(1/4)^2}=\frac{\sqrt{144+25}}{20}=\frac{13}{20}\,.$$

Therefore $d=3/13$.

Problem 5 Answers: parallel for $k=\pm\sqrt{8/3}$; not the same line for any value of k; for $k\neq\pm\sqrt{8/3}$ the two lines meet at

$$\Big(\frac{12}{8-3k^2}\,,\,\frac{9k}{8-3k^2}\Big)\,.$$

Problem 6 Answers: $(x+3)^2+(y-9)^2=25$; outside; inside; on

Problem 7 Answers for two parts: center at $(-\frac{5}{2},3)$ and radius equals $\frac{1}{2}\sqrt{21}$; points of intersection are $(-\frac{1}{2}+\frac{\sqrt{5}}{10}\,,\,2+\frac{\sqrt{5}}{5})$ and $(-\frac{1}{2}-\frac{\sqrt{5}}{10}\,,\,2-\frac{\sqrt{5}}{5})$

Problem 8 Place the origin at one vertex and orient the axes so that the two sides of the rectangle emanating from that vertex lie on the positive x-axis and the positive y-axis—with other endpoints, say, at $(a, 0)$ and $(0, b)$. Since opposite sides of the rectangle are parallel and lines parallel to the x-axis have the form $y =$ (a constant) and lines parallel to the y-axis have the form $x =$ (a constant) the fourth vertex is at (a, b). The length of the diagonal from $(0, 0)$ to (a, b) equals

$$\sqrt{(a-0)^2 + (b-0)^2} = \sqrt{a^2 + b^2}\,.$$

The length of the other diagonal equals

$$\sqrt{(0-a)^2 + (b-0)^2} = \sqrt{a^2 + b^2}\,.$$

The two lengths are the same, as desired.

Problem 9 Solution: Consider a line k that is tangent to the circle and passes through $(0, b)$ (if there is such a line). Let (s, t) denote the point of tangency. Then

$$s^2 + t^2 = 3$$

since (s, t) is on the circle; and, provided that neither s nor t equals 0,

$$\frac{t-b}{s-0} = -\frac{1}{(t-0)/(s-0)}$$

since the line through $(0, b)$ and (s, t) is perpendicular to the line through $(0, 0)$ and (s, t). From the second of these two equations it follows that $s^2 = -t^2 + bt$. Substitution for s^2 in the first of the two equations gives

$$bt = 3\,,$$

for which it follows that $t = 3/b$ and

$$s^2 = -\frac{9}{b^2} + 3\,.$$

In order that s^2 be nonnegative, $|b|$ must be greater than or equal to $\sqrt{3}$. So there is no tangent line through $(0, b)$ if $|b| < \sqrt{3}$. If $|b| > \sqrt{3}$, there are two possible values of s —namely,

$$\pm\frac{1}{|b|}\sqrt{3b^2 - 9}\,.$$

The slopes of the two lines are given by

$$\frac{t-b}{s-0} = \frac{(3/b) - b}{\pm|b|^{-1}\sqrt{3b^2-9}} = \mp\frac{|b|}{b\sqrt{3}}[b^2 - 3]^{1/2} = \mp\frac{|b|}{b}\sqrt{\frac{b^2-3}{3}}\,,$$

where $\mp$ has been used in order to identify the minus sign with the plus sign in $\pm$. The equations of the two lines are

$$y = \mp\frac{|b|}{b}\sqrt{\frac{b^2-3}{3}}\,x + b$$

It remains for us to consider $|b| = \sqrt{3}$ as well as the cases where either s or t equals 0. The only way that the point of tangency can have the form $(0, t)$ for some t is that $(0, b)$ be the point of tangency. This happens when $(0, b)$ is on the circle—that is, $|b| = \sqrt{3}$ —in which case there is one tangent line—namely, $y = b$.

Finally, we consider $t = 0$. The line from $(0, 0)$ to either of the points $(\pm\sqrt{3}, 0)$ is horizontal so any tangent line through either of $(\pm\sqrt{3}, 0)$ would have to be vertical and could not, therefore, pass through $(0, b)$.

[Comment: The conclusion that there: (i) is no tangent line when $|b| < \sqrt{3}$, (ii) is one tangent line when $|b| = \sqrt{3}$, and (iii) are two tangent lines when $|b| > \sqrt{3}$ is consistent with the fact that $(0, b)$ is in the interior of the circle when $|b| < \sqrt{3}$, is on the circle when $|b| = \sqrt{3}$, and is exterior to the circle when $|b| > \sqrt{3}$.]

Problem 10 Solution:

$$|x + 2| = \sqrt{(x + 5)^2 + (y - 0)^2}\,.$$

Since both sides are nonnegative, we introduce no extraneous solutions by squaring both sides:

$$x^2 + 4x + 4 = x^2 + 10x + 25 + y^2\,,$$

which simplifies to

$$6x = -y^2 - 21\,,$$

the equation of some parabola.

Problem 11 Description of picture; answer: The picture is that of a pentagon together with its interior. The vertices of the pentagon are at $(0, 1)$, $(0, 0)$, $(4, 0)$, $(\frac{5}{2}, \frac{3}{2})$, and $(\frac{5}{4}, \frac{9}{4})$; maximum value is 12

Problem 12 Distance: 340

Problem 13 Answer: $\frac{5}{2}\sqrt{3}$

Chapter 8

English into Mathematics

Problem 1 Answers: $0.34\,x + 1.61\,y$; 5

Problem 2 Answers: $1.06(0.34\,x+1.61\,y)$ or $(0.3604\,x+1.7066\,y)$ rounded to the nearest penny (assuming that is how the sales tax works in that state); 4

Problem 3 Answer: 56 years (and no fraction)

Problem 4 Solution: Let w denote the width of the rectangle and x the side length of the square. Then $x^2 = w(w+3)$, so $x = \sqrt{w(w+3)}$.
Suppose that $x = 7$. Then $w(w+3) = x^2 = 49$. So we need to solve

$$w^2 + 3w - 49 = 0\,.$$

The positive solution is

$$\frac{-3+\sqrt{205}}{2}\,.$$

This is the width of the rectangle. We get its length by adding 3:

$$\frac{3+\sqrt{205}}{2}\,.$$

Problem 5 Answers: $312\frac{1}{2}$ square meters; less than [Comment: The problem indicates that a calculator might be useful. Actually, all that one needs is that $4 > \pi$.]

Chapter 9

Speed and Distance

Problem 1 Answer: 75 miles

Problem 2 Answer: 48 miles per hour [Comment: An intuitive explanation of why the answer is less than the numerical average of 40 and 60 is that Jeremiah spends more time traveling at 40 miles per hour than he does at 60 miles per hour.]

Problem 3 180 kilometers (not to be confused with the 180 kilometers that each has traveled)

Problem 4 Answers: $a = 50$ feet per second and $b = 200$ feet; $(25 + 5\sqrt{153})/16$ seconds which to nearest tenth of a second equals 5.4 seconds; The negative time indicates how many seconds in the past the object would have had to have left the ground (at an appropriate speed) in order to follow the same path as the given object once time would become positive; $10\sqrt{153}$ feet per second which is approximately 1.2×10^2 feet per second to two significant digits

Chapter 10

Trigonometric Simplification

Problem 1 Answers: $\cos\theta$; $\tan^2\varphi$; $(1+5\cos\psi)^2$

Problem 2 Solution: Since $90^\circ < \theta < 180^\circ$, we see that $\cos\theta < 0$. We use this fact and various trigonometric identities to obtain:

$$\begin{aligned}
\cos\theta &= -\sqrt{1-(9/17)} = -\sqrt{8/17} = -2\sqrt{2/17}\,;\\
\tan\theta &= (\sin\theta)/(\cos\theta) = -3/\sqrt{8} = -3/(2\sqrt{2}\,)\,;\\
\cot\theta &= 1/\tan\theta = -2\sqrt{2}\,/3\,;\\
\sec\theta &= 1/\cos\theta = -\sqrt{17/8} = -\tfrac{1}{2}\sqrt{17/2}\,;\\
\csc\theta &= 1/\sin\theta = \sqrt{17}\,/3\,.
\end{aligned}$$

[Comment: Notice that the above solution did not involve finding θ.]

Problem 3 Answer: 5.076992 radians

Problem 4 Solution: In the double-angle formula replace 2θ by φ to obtain

$$\cos\varphi = 1 - 2\sin^2(\varphi/2)\,.$$

Now rearrange:

$$2\sin^2(\varphi/2) = 1-\cos\varphi\,.$$

Divide by 2 and take square roots on both sides:

$$\sin(\varphi/2) = \pm\sqrt{\frac{1-\cos\varphi}{2}}\,.$$

Both signs need to be considered. For some φ the positive sign is correct; for other φ the negative sign is correct. But for any particular φ only one of the two signs is correct, except when $\cos\varphi = 1$ in which case the sign is irrelevant.

Problem 5 Solution: In view of the next problem, in the 'solution' here we will not concern ourselves with the 'division by zero' issue. Thus this 'solution' is not complete since in the following calculation we divide both numerator and denominator by $\cos\alpha\cos\beta$:

$$\begin{aligned}\tan(\alpha+\beta) &= \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \frac{\sin\alpha\cos\beta+\sin\beta\cos\alpha}{\cos\alpha\cos\beta-\sin\alpha\sin\beta}\\ &= \frac{\frac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta}+\frac{\sin\beta\cos\alpha}{\cos\alpha\cos\beta}}{\frac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta}-\frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}} = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}\,.\end{aligned}$$

Problem 6 Solution: Suppose that $\tan\alpha$ is undefined but that $\tan\beta$ is defined. Then $\cos\alpha = 0$ and $\cos\beta \neq 0$. In this case the formulas for $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$ given in the preceding problem simplify:

$$\begin{aligned}\sin(\alpha+\beta) &= \sin\alpha\cos\beta\,;\\ \cos(\alpha+\beta) &= -\sin\alpha\sin\beta\,.\end{aligned}$$

The quotient equals $-\cot\beta = 1/(-\tan\beta)$. This is also what one would get from the formula

$$\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

were one to divide both numerator and denominator by $\tan\alpha$ and then interpret $1/\tan\alpha$ as 0 when $\tan\alpha$ is undefined.

A similar interpretation is appropriate when $\tan\alpha$ is defined but $\tan\beta$ is undefined.

Suppose that both $\tan\alpha$ and $\tan\beta$ are undefined. Then $\cos\alpha = \cos\beta = 0$. From the formula for $\sin(\alpha+\beta)$ we then get $\sin(\alpha+\beta) = 0$ from which it follows that $\tan(\alpha+\beta) = 0$. One can view this value of 0 as arising from the formula

$$\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

in the following manner. Divide both numerator and denominator by $\tan\alpha\tan\beta$; then interpret both $1/\tan\alpha$ and $1/\tan\beta$ as 0 if $\tan\alpha$ and $\tan\beta$ are both undefined.

Now suppose that $1-\tan\alpha\tan\beta = 0$, in which case $\cos\alpha\cos\beta \neq 0$, because $\tan\alpha$ and $\tan\beta$ are defined. Multiply both sides of $1-\tan\alpha\tan\beta = 0$ by $\cos\alpha\cos\beta$ to obtain

$$\cos\alpha\cos\beta-\sin\alpha\sin\beta = 0\,.$$

That is, $\cos(\alpha+\cos\beta) = 0$ in which case $\tan(\alpha+\beta)$ is undefined.

Problem 7 Answers: $\frac{\pi}{2}$; $\frac{\pi}{3}$; $\frac{1}{2}$; $\sqrt{3}/2$

Problem 8 Solution: Multiply the numerator and denominator of the first term by $1+\sin\varphi$ and the numerator and denominator of the second term by $\cos\varphi$. We get

$$\frac{\cos\varphi(1+\sin\varphi)}{\cos^2\varphi}+\frac{(1-\sin\varphi)\cos\varphi}{\cos^2\varphi}\,,$$

which equals $2/\cos\varphi = 2\sec\varphi$. [Comment: Since the given expression is meaningless if $\cos\varphi = 0$ we are permitted to assume $\cos\varphi \neq 0$ from which it follows that $1-\sin\varphi \neq 0$ and $1+\sin\varphi \neq 0$. Thus: (i) the original formula is defined for exactly those φ for which $\cos\varphi \neq 0$ —and the same is true of the answer $2\sec\varphi$; (ii) nowhere did we multiply both numerator and denominator of a fraction by something that might equal 0.]

Problem 9 Solution: A complete solution should contain a proof that the diagonals of a rectangle inscribed in a circle meet in the center of the circle. This part of the solution will not be given here.

Let θ denote the measure of the angle at which the two diagonals meet. The two diagonals partition the rectangle into four isosceles triangles each of which has two sides of length 3. The vertex angle of two of these triangles has measure θ and the vertex angle of the other two triangles has measure $\pi - \theta$. Therefore the heights of the triangles are $3\cos\frac{\theta}{2}$ and $3\cos\frac{\pi-\theta}{2} = 3\sin\frac{\theta}{2}$. By the Pythagorean Theorem, the bases have lengths

$$2\sqrt{9 - 9\cos^2(\theta/2)} = 6\sin(\theta/2)$$

and

$$2\sqrt{9 - 9\sin^2(\theta/2)} = 6\cos(\theta/2)$$

Thus all four of the triangles have the same area:

$$\tfrac{1}{2}\cdot 6\sin(\theta/2)3\cos(\theta/2) = \tfrac{9}{2}\sin\theta\,.$$

The area of the rectangle is $18\sin\theta$.

Chapter 11

Trigonometric Equations

Problem 1 Solution sets:

$\{\frac{\pi}{2} + n\pi\colon n \text{ an integer}\} \cup \{\frac{\pi}{6} + 2n\pi\colon n \text{ an integer}\} \cup \{\frac{5\pi}{6} + 2n\pi\colon n \text{ an integer}\}$;

$\{\pm \arccos(\frac{\sqrt{5}-1}{2}) + 2n\pi\colon n \text{ an integer}\}$

Problem 2 Solution set:

$$\{\pm \arccos(\tfrac{\sqrt{33}-3}{4}) + 360^\circ n\colon n \text{ an integer}\}\,,$$

with the understanding that arccos is to be interpreted in degrees (not radians). The nearest-minute approximations of the solutions are

$$\pm 46^\circ 40' + 360^\circ n\,.$$

Problem 3 Answer: none if $|k| < 1$, one if $|k| = 1$, two if $|k| > 1$

Problem 4 Answers: By using the Law of Cosines, we get that the angles opposite the sides of lengths 7, 10, and 14 have measures

$$\arccos(247/280)\,, \quad \arccos(145/196)\,, \quad \arccos(-47/140)\,,$$

respectively. Their nearest-degree approximations are 28°, 42°, and 110°.

Problem 5 Solution: The area equals $5{\cdot}8{\cdot}\sin(1.3)$ which is approximately 38.542. The lengths of the two diagonals can be obtained immediately from the Law of Cosines:

$$\sqrt{5^2 + 8^2 \mp 2 \cdot 5 \cdot 8\cos(1.3)} = \sqrt{89 \mp 80\cos(1.3)}\,.$$

where the $\mp$ arises from the facts that other angles of the parallelogram have measure $\pi - 1.3$ radians and $\cos(\pi - 1.3) = -\cos(1.3)$. The three-decimal-place approximations are 8.222 and 10.507.

Partial check using the fact that in a parallelogram the sum of the squares of the lengths of the two diagonals equals the sum of the squares of the four sides, which in this case equals 178:

$$8.222^2 + 10.507^2 = 177.998\,.$$

Seems ok.

Problem 6 Solution A: Applying the Law of Cosines to a triangle two of whose sides are sides of the pentagon and one of whose sides is a diagonal, we obtain

$$d^2 = x^2 + x^2 - 2xx\cos(3\pi/5)\,.$$

Thus,

$$x = \frac{d}{\sqrt{2 - 2\cos(3\pi/5)}}\,.$$

An approximation of this relation is $x = 0.618\,d$.

Solution B: We use the Law of Sines in the same isosceles triangle used above in Solution A. The two congruent angles in that triangle each have measure $[\pi - (3\pi/5)]/2 = \pi/5$. Thus,

$$\frac{x}{\sin(\pi/5)} = \frac{d}{\sin(3\pi/5)}\,,$$

from which it follows that

$$x = \frac{\sin(\pi/5)d}{\sin(3\pi/5)} = \frac{\sin(\pi/5)d}{\sin(\pi - (3\pi/5))} = \frac{\sin(\pi/5)d}{\sin(2\pi/5)} = \frac{\sin(\pi/5)d}{2\sin(\pi/5)\cos(\pi/5)} = \frac{1}{2}\sec(\pi/5)d\,.$$

[Direct proof that the two solutions are equal, using a double-angle formula:

$$\frac{1}{\sqrt{2 - 2\cos^2(3\pi/5)}} = \frac{1}{\sqrt{2 - 2(1 - 2\sin^2(3\pi/10))}} = \frac{1}{2\sin(3\pi/10)} = \frac{1}{2\cos((\pi/2) - (3\pi/10))} = \frac{1}{2\cos(\pi/5)} = \frac{\sec(\pi/5)}{2}\,.]$$

[Exact value of $\sec(\pi/5)$: Clever manipulation with double-angle formulas followed by solving a fourth degree polynomial equation gives the value of $\cos(\pi/5)$ and thus the value of $\sec(\pi/5)$ —namely, $\sec(\pi/5) = \sqrt{5} - 1$ from which it follows that

$$x = \frac{\sqrt{5} - 1}{2}\,d\,.$$

I have not included such a manipulation in the problem list because it relies too heavily on ingenuity. The purpose of the problem list is to emphasize relevant knowledge, understanding, and basic skills, together with the ability to carefully organize and write solutions of several-step problems.]

[A comment on geometric constructions: The fact that $\sec(\pi/5)$ can be expressed in terms of integers and their square roots is related to the fact that it is possible to construct a regular pentagon with straightedge and compass.]

Chapter 12

Functions

Problem 1

Problem 2

Problem 3 Answers: $(f \circ h)(x) = 4x^2 + 20x + 22\,;$ $(g \circ f)(v) = \sqrt{v^2 + 2}\,;$ $(f \circ g)(x) = x + 2\,,\ x \geq -5\,;$ $h^{-1}(y) = \frac{1}{2}y - \frac{5}{2}\,;$ $g^{-1}(u) = u^2 - 5\,,\ u \geq 0$

Problem 4 Move it to the left 2 units and up 3 units (presuming that, as is usual, the positive axis for the domain points rightward and the positive axis for the values of the function points upward.

Problem 5 Solutions: Part (i): The domain is obviously the entire real line. For $f(x)$ to be positive, x must be different from 2, -3 and 5 —and, moreover, an even number of the factors (counting the initial factor '-5') must be negative. Thus, one of $(x + 3)$ and $(x - 5)$ must be negative and the other must be positive. The x's for which this is true are those between -3 and 5. Therefore, $f(x) > 0$ if and only if $-3 < x < 2$ or $2 < x < 5$.

Part (ii): Clearly, $\sqrt{f(x)} > 0$ if and only if $f(x) > 0$. By Part (i) this is true if and only if $-3 < x < 2$ or $2 < x < 5$. In addition to these x's, those for which $f(x) = 0$ are in the domain of g. Hence, the domain of g equals the closed interval $[-3, 5]$.

Part (iii): The domain consist of those x for which both $(3 - x)$ and $(3 + x)$ are positive—that is, the domain is the open interval $(-3, 3)$. For x in this interval, the given function equals $\log_{10}(9 - x^2)$. It is positive if and only if $9 - x^2 > 1$, which is true if and only if $-\sqrt{8} < x < \sqrt{8}$. [Comment: One might prefer to write $2\sqrt{2}$ instead of $\sqrt{8}$.]

Part (iv): The domain of arccos is the closed interval $[-1, 1]$. Accordingly, we want to solve the inequality

$$-1 \leq \frac{x + 2}{x + 1} \leq 1\,.$$

Case (a): $x > -1$: In this case the above inequality is equivalent to

$$-(x+1) \leq x+2 \leq (x+1)\,.$$

The rightmost inequality never holds.

Case (b): $x < -1$: Now the inequality of interest is equivalent to

$$-(x+1) \geq x+2 \geq (x+1)\,.$$

The rightmost inequality always holds, and the leftmost inequality is equivalent to $-3 \geq 2x$, or $x \leq -3/2$. All x's which satisfy this inequality belong to this case.

We conclude that the domain of the given function is the interval $(-\infty, -3/2]$.

In order for the given function to equal 0, $(x+2)/(x+1)$ would have to equal 1. This does not happen for any x. We combine this fact with the fact that all values of arccos belong to the interval $[0, \pi]$ to conclude that the given function is positive for all values in its domain.

Problem 6 Answers: ± 2; $x = \pm 1$; none; $y = 1$; one-to-one on each of the following four intervals: $(-\infty, -1)$, $(-1, 0]$, $[0, 1)$, and $(1, \infty)$. Here as functions of y are formulas for the respective inverses:

$$\begin{aligned}
-\sqrt{(y-4)/(y-1)}\,, &\quad y < 1\,,\\
-\sqrt{(y-4)/(y-1)}\,, &\quad y \geq 4\,,\\
\sqrt{(y-4)/(y-1)}\,, &\quad y \geq 4\,,\\
\sqrt{(y-4)/(y-1)}\,, &\quad y < 1
\end{aligned}$$

Problem 7 Answer: $y = \sqrt{r^2 - x^2}$ and $y = -\sqrt{r^2 - x^2}$

Problem 8 Answer:

$$g^{-1}(y) = \frac{\sin y}{1 - \sin y}\,, \qquad -\frac{\pi}{2} \leq y < \frac{\pi}{2}$$

Problem 9 Solution: We use the formula for the cosine of a sum:

$$\begin{aligned}
\cos 3x &= \cos 2x\, \cos x - \sin 2x\, \sin x\\
&= (2\cos^2 x - 1)\cos x - 2\sin x\, \cos x\, \sin x\\
&= 2\cos^3 x - \cos x - 2\cos x(1 - \cos^2 x)\\
&= 4\cos^3 x - 3\cos x\,.
\end{aligned}$$

So the answer is $p(u) = 4u^3 - 3u$.

Problem 10

Problem 11 Answer:

$$\{0\} \cup \{x\colon (2n-1)\pi \leq |x| \leq 2n\pi \text{ for some positive integer } n\}$$

Problem 12 Answer:

$$3x^2 + 3xh + h^2\,, \quad h \neq 0$$

Chapter 13

Solid Geometry

Problem 1 Answer: $d = \sqrt{3}\,x$, where x denotes the side length and d denotes the length of a long diagonal.

Problem 2 Answers: 8, 4

Problem 3 Answers: $r = (\frac{3V}{4\pi})^{1/3}$, where V denotes the volume and r the radius; 1 foot, $9\frac{6}{8}$ inches

Problem 4 Answer: Reduction of the fraction to one with a denominator 4 can give the impression that the answer is only guaranteed to be correct to the nearest quarter of an inch. For instance if $\frac{6}{8}$ had been replaced by $\frac{3}{4}$ in the preceding answer, one might think that all that were being claimed is that the actual value is between 1 foot, $9\frac{5}{8}$ inches and 1 foot, $9\frac{7}{8}$ inches; whereas the answer as written indicates a claim that the answer is between 1 foot, $9\frac{11}{16}$ inches and 1 foot, $9\frac{13}{16}$ inches.

Problem 5 Answers: yes; no; yes

Problem 6 Answers: $27\pi/2$ square feet; 42 square feet

Problem 7 Solution: One approach is to focus on a right triangle whose hypotenuse is a slanting edge of one the triangular faces and whose right angle is at the center of the base of the pyramid. The hypotenuse of this triangle has length 2 feet. The horizontal edge of this right triangle has length $\sqrt{2}$ feet since its length is half the length of a diagonal of a face of the cube. By the Pythagorean Theorem its vertical edge has length $\sqrt{2^2 - (\sqrt{2})^2}$ feet, which equals $\sqrt{2}$ feet.

From the previous paragraph we deduce that the height of the square pyramid equals $\sqrt{2}$ feet and thus that the volume of this pyramid is $\frac{1}{3} \cdot 2^2 \cdot \sqrt{2}$ cubic feet. Since the volume of the cube is 8 cubic feet, we get that the total volume equals

$$8 + \frac{4\sqrt{2}}{3} \quad \text{cubic feet.}$$

Since the triangle in the first paragraph of this solution is an isosceles triangle, we get that the angle of interest has measure $90^\circ + 45^\circ = 135^\circ$.

Problem 8 Answers: 8790 miles at 45° latitude and 6216 miles over the North Pole, both distances being approximated to the nearest mile.

Problem 9 Answer: a 180° rotation about an axis that is concurrent with the axes in the problem and perpendicular to them.

Problem 10

Problem 11 Answer:

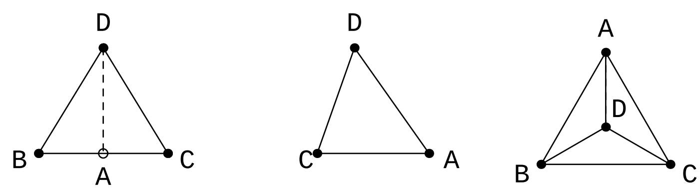

Figure 13.1: Problem 11: front, side and top views

I will take the edge lengths of the tetrahedron to be 6, $\triangle ABC$ to be the bottom face of the tetrahedron, and edge BC to go from left-to-right as viewed from the front.

In each of the three views of the tetrahedron the lines that we see can be regarded as the projected images of the edges onto a vertical plane which is perpendicular to the line of sight.

In the front view, shown at the left, the full length of the segment $\overline{BC}$ projects accurately as having length 6. The projections of the edges $\overline{BD}$ and $\overline{CD}$ have length $\sqrt{33}$, and the projection of the hidden edge $\overline{AD}$ has length $\sqrt{24}$.

In the side view, shown in the middle, vertex B is directly behind vertex C, edge $\overline{AD}$ projects to its full length of 6, and the projections of edges $\overline{CA}$ and $\overline{CD}$ both have length $\sqrt{27}$.

In the top view, shown at the right, the edges $\overline{CA}$, $\overline{AB}$, and $\overline{BC}$ of the base show at full length 6, and the other three edges project as having length equal to 2/3 the length of a median of $\triangle ABC$ —namely, $\sqrt{12}$.

Description of planar representation: Assume the tetrahedron is transparent and stand it on a vertex—call that vertex D —on a flat surface, so that the face $\triangle ABC$ opposite it is horizontal. (Thus, with reference to the figure above, the top view has become a bottom view.) Place a light a bit above the centroid of the top face and look at the shadows of the edges on the surface on which D sits. The resulting picture is similar to the top view described in the preceding paragraph, except that if the same scale were to be used, the picture of the shadow would be much larger than the picture of the top view above. The three smaller triangles in the picture represent three of the four faces of the tetrahedron. In order to have the number of regions in the plane equal the number of faces, we associate the exterior region in the planar picture with the top face $\triangle ABC$.

Problem 12

Chapter 14

Discrete Mathematics

Problem 1 Solution: There are 9 choices for the leftmost digit (since '0' is not a possible leftmost digit, then 9 choices for the second digit, 8 choices for the third digit, 7 choices for the fourth digit, and only 6 choices for the last digit (in order that it be different from each of the first four digits). The answer is

$$9 \cdot 9 \cdot 8 \cdot 7 \cdot 6 = 27216\,.$$

Problem 2 Solutions:

Part (i): $5 + 7$ counts all those who are sufficiently tall or sufficiently heavy, but double-counts the three who are both. Correcting for the double-count we get the answer: $5 + 7 - 3 = 9$.

Part (ii): The picture will not be drawn here, but it can be drawn using two interlocking circles with, say, the interior of the left-hand circle representing the students who weigh over 160 pounds and the interior of the right-hand circle representing the students who are at least $5'11$" tall. Then the number 3 should be placed in the intersection of the two interiors, the number 2 in the intersection of the interior of the left-hand circle and the exterior of the right-hand circle, the number 4 in the intersection of the interior of the right-hand circle and the exterior of the left-hand circle, and the number 11 in the intersection of the two exteriors. The numbers 2 and 4 in the preceding sentence are obtained by subtracting 3 from 5 and 7, respectively. The number 11 is chosen so as to make the sum of the four numbers equal to 20.

Part (iii): For this part we are to count exactly those students who did not qualify to be counted in Part (i). Thus the answer for this part is $20 - 9 = 11$. This answer could also be obtained immediately from the Venn diagram of Part (ii).

Problem 3

Problem 4 Preface: There are different symbols for the choose symbol. I will use

$$\binom{n}{k}$$

to denote the number of ways of choosing k objects from a set of n distinct objects.

Solution: The number of ways of choosing the two men equals $\binom{20}{2}$ and the number of ways of choosing the two women equals $\binom{14}{2}$. The answer is

$$\binom{20}{2}\binom{14}{2} = \frac{20 \cdot 19}{2 \cdot 1} \cdot \frac{14 \cdot 13}{2 \cdot 1} = 190 \cdot 91 = 17290\,.$$

Problem 5 Solution: After doing some arithmetic one arrives at the prime factorizations:

$$\begin{aligned} 3150 &= 2 \cdot 3^2 \cdot 5^2 \cdot 7\,, \\ 5070 &= 2 \cdot 3 \cdot 5 \cdot 13^2\,. \end{aligned}$$

A step that some find helpful is to rewrite these prime factorizations so that the same primes appear explicitly in each line and every prime has an exponent:

$$\begin{aligned} 3150 &= 2^1 \cdot 3^2 \cdot 5^2 \cdot 7^1 \cdot 13^0 \\ 5070 &= 2^1 \cdot 3^1 \cdot 5^1 \cdot 7^0 \cdot 13^2\,. \end{aligned}$$

To get the greatest common divisor we use the smaller of the exponents for each prime:

$$2^1 \cdot 3^1 \cdot 5^1 \cdot 7^0 \cdot 13^0 = 30\,.$$

To obtain the least common multiple we use the larger of the exponents for each prime:

$$2^1 \cdot 3^2 \cdot 5^2 \cdot 7^1 \cdot 13^2 = 532350\,.$$

Problem 6 Solution: An arbitrary odd integer can be written in the form $2n+1$ for some integer n. The square of this integer equals

$$(2n+1)^2 = 4n^2 + 4n + 1 = 4(n^2+n) + 1\,.$$

Since $4(n^2+n)$ is divisible by 4, the quantity $4(n^2+n)+1$ will give a remainder of 1 when divided by 4.

Moreover, since $4(n^2+n) = 4n(n+1)$ and either n or $n+1$ is divisible by 2, it follows that $4(n^2+n)$ is divisible by 8 and thus that $4(n^2+n)+1$ will give a remainder of 1 when divided by 8.

[Comment: Some thought has to be given to the meaning of a positive remainder when a negative integer is divided by a positive integer. When an appropriate interpretation is given, the above proof is valid whether the odd integer is positive or negative.]

Chapter 15

Probability and Statistics

Problem 1 Answers and Solution: 1/32; 1/2; 1/2

Part (iv), Method A: There are 32 possible sequences of which two have the property that no two consecutive flips have the same result—namely, HTHTH and THTHT. The answer is $\frac{2}{32} = \frac{1}{16}$ [Comment: It is legitimate to do this problem via counting as we have just done, since the coin is fair.]

Part (iv), Method B: Any result on the first flip is ok. But thereafter, the probability of any particular flip being satisfactory is 1/2. Thus the answer is $(1/2)^4 = 1/16$.

Problem 2 Answers and Solution: 1/10; 1/190; 4/190; 1/10

Part (v): Replace the cards labeled by 2 and 6 with four blank cards. Now shuffle the deck randomly. Then randomly choose two of the blank cards on which to write 2 and then write 6 on the other two blank cards. This experiment is equivalent to the experiment described in the problem. There are a total of

$$\frac{4!}{2!\,2!} = 6$$

ways of choosing the pair of blank cards on which to write 2. For only one choice will it result in the two cards with the label 2 being above the two cards with the label 6. Therefore, the answer is 1/6.

Problem 3 Answer: 0.16

Problem 4 Answer: $y = -1.56x + 6.69$, where x denotes the first coordinate and y denotes the second.

Chapter 16

Listening to Discussions about Mathematics

First Conversation:

$$73 \cdot 87 = (80 - 7)(80 + 7) = 80^2 - 7^2 = 6400 - 49 = 6351\,.$$

Second Conversation:

$$\begin{array}{r} 37 \\ \times\underline{37} \\ 259 \\ \underline{1110} \\ 1369 \end{array}$$

$(1369 - 1) \div 2 = 684$. So the three side lengths are 37, 684, and 685.

Chapter 17

Pre-Seventh Grade; Later Reinforcement Within Math

Problem 1 Answers: $0.\overline{714285}$; $\overline{82.35294117647058}$; $0.2\overline{523809}$. The bar indicates the repeating portion.

Problem 2 Answers: 35880/99; 519/990. If one desires lowest terms, the answers are 11960/33 and 173/330.

Problem 3

Problem 4 Answer: 10 yards, 1 foot, 7 inches

Problem 5 Solution: We find from a table that one mile is approximately 1.60935 kilometers which is the same as 1609.35 meters. Then we do a lot of multiplying by 1 in various guises:

$$\frac{50 \text{ miles}}{\text{hour}} \cdot \frac{1 \text{ hour}}{60 \text{ minutes}} \cdot \frac{1 \text{ minute}}{60 \text{ seconds}} \cdot \frac{1609.35 \text{ meters}}{1 \text{ mile}} = 22.35 \frac{\text{meters}}{\text{seconds}},$$

correct to four significant digits. The answer is 22.35 meters per second.

Problem 6 Answers: 87, 77, $2\frac{1}{2}$, $3\frac{2}{5}$

Problem 7 Solution: If the par on every hole were 4, par for the course would be $18 \times 4 = 72$. The par 5 hole in combination with one of the par 3 holes is like two part 4 holes. The remaining par 3 hole decreases par for the course by 1. The answer is 71.

Problem 8 Answers: $22\frac{17}{44}$, $1\frac{5}{6}$

Problem 9 Answers: $\frac{-229}{48}$, $\frac{50}{203}$, $\frac{1}{16}$

Problem 10 22802763/3549301

Problem 11

Problem 12 Solution: The vertical line separating the front wall from the left side wall never meets the horizontal line separating the right side wall from the floor. These two lines are not parallel because they do not even lie in a common plane.

Problem 13 Answer: two

Problem 14 Definition: A pentahedron is a three-dimensional figure with five flat faces. Alternatively, one could say it is a five-faced polyhedron.

One common pentahedron is a square pyramid. One of its faces, usually called the *base,* is a square. The other faces are triangles each of which has one edge in common with the base.

Another common pentahedron is a triangular prism. It consists of two congruent triangles, one lying directly above the other, with three rectangular faces formed by edges connecting corresponding vertices of the two triangles.

Problem 15 Answers: 40 per cent; 67 per cent; 38 per cent. The last answer could also be 37 per cent since the exact answer is 37.5 per cent; the problem did not give an instruction for resolving a tie when doing the rounding-off.

Problem 16 Descriptive response for Part (iii): front view is a square, top and side views are congruent rectangles which do not happen to be squares but which have two sides of the same length as the sides of the square in the front view.

Chapter 18

Pre-Seventh Grade; Later Reinforcement Outside of Math

The main point of this chapter is that students should, by the time they finish sixth grade: (i) have a sense of size and how it it is represented by numbers when discussing various topics that are not typically regarded as mathematics; (ii) be able to organize data in a variety of ways. Therefore, for this chapter, even more than for the preceding chapters, it should be remembered that the particular problems are not special in their importance. For instance, some knowledge about liquid measure or the amount of honey produced by a hive of bees in an apiary is just as good as the knowledge highlighted by some of the questions in this chapter. The possibilities are endless—and it would be a mistake to try to be anywhere near comprehensive. To do so would result in deemphasizing important topics in mathematics itself.

I make some further comments in order to clarify my view. When a student is taking, say, geography in middle school and the teacher is either making a pie chart or using numbers to help the students understand the population distribution in some country or continent, it is important that the pie chart and the numbers actually be an aid for the student, not an additional hurdle. General familiarity with data and size is something I would like a student coming out of sixth grade mathematics to have, but it is not important for the student to have developed this familiarity in a wide variety of contexts. The various middle school and high school subjects outside of mathematics can provide such a variety.

Problem 1 Answer: $250,000$

Problem 2 Answer: 4×10^5

Problem 3 Answers: 23° South; December 21; Winter

Problem 4 Answer: 35%

Problem 5 Answer: Woodrow Wilson

Problem 6 Answer: $90,000$

Problem 7 Answer: Tuesday

Problem 8

Problem 9