Algebra Prelim Written Exam Fall 2011

Questions are equally weighted. Give essential explanations and justifications: a large part of each question is demonstration of understanding the context and of which issues are primary. Do not choose assumptions or contexts making the problems silly. Coherent, legible writing is essential: your paper should not be a puzzle for the grader.

Write your codename, not actual name, on each booklet. No notes, books, calculators, computers, cell phones, wireless, bluetooth, or other communication devices may be used during the exam.

[1] Describe all abelian groups of order 72.


[3] Find all pairs of integers $x, y$ such that $157x + 101y = 1$.

[4] Let $A$ be a finite abelian group of $\mathbb{C}$-linear maps of a finite-dimensional $\mathbb{C}$-vectorspace $V$ to itself. Prove that there is at least one simultaneous eigenvector.

[5] Prove that the ideal generated by 7 and $x^2 + 1$ is a maximal ideal in $\mathbb{Z}[x]$.

[6] Describe all intermediate fields between $\mathbb{Q}$ and $\mathbb{Q}(\zeta_9)$, where $\zeta_9$ is a primitive ninth root of unity.

[7] Prove that the tenth cyclotomic polynomial

$$\Phi_{10}(x) = \frac{(x^{10} - 1)(x - 1)}{(x^5 - 1)(x^2 - 1)} = x^4 - x^3 + x^2 - x + 1$$

is irreducible in $\mathbb{F}_3[x]$, where $\mathbb{F}_3$ is the finite field with 3 elements.