Questions are equally weighted. Give essential explanations and justifications: a large part of each question is demonstration that you understand the context and understand which issues are primary. Do not choose assumptions or contexts making the problems silly. Coherent writing is essential: your paper should not be a puzzle for the grader.

Write your codename, not actual name, on each booklet. No notes, books, calculators, computers, cell phones, wireless, bluetooth, or other communication devices may be used during the exam.

[1] Prove that every abelian group of order 105 is cyclic.

[2] Let $p < q$ be primes, with $q \neq 1 \mod p$. Show that every group of order $pq$ is abelian.

[3] Prove that the polynomial $x^5 + y^5 + 1$ is irreducible in $\mathbb{C}[x, y]$.

[4] Let $S, T$ be $\mathbb{C}$-linear maps of a finite-dimensional $\mathbb{C}$-vectorspace $V$ to itself, with $ST = TS$. Prove that $S, T$ have at least one common eigenvector.

[5] For distinct prime numbers $p_1, \ldots, p_n$, show that there are integers $a_1, \ldots, a_n$ such that

$$\frac{1}{p_1 \ldots p_n} = \frac{a_1}{p_1} + \ldots + \frac{a_n}{p_n}$$

[6] Describe all intermediate fields between $\mathbb{Q}$ and $\mathbb{Q}(\zeta_{12})$, where $\zeta_{12}$ is a primitive twelfth root of unity.

[7] Let $G$ be the group of invertible 2-by-2 matrices over the field $\mathbb{F}_7$ with 7 elements. Find a 3-Sylow subgroup of $G$. 