Questions are equally weighted. Give essential explanations and justifications: a large part of each question is demonstration of understanding of the context and of which issues are primary. Do not make assumptions or choose contexts making the problems silly. Coherent writing is essential: your paper should not be a puzzle for the grader.

Put your codename, not your actual name, on your paper. Please follow the rules for this exam, as explained by the proctor and by the office of the Director of Graduate Studies in Mathematics.

[1] Let $H$ be an index-two subgroup of a finite group $G$. Show that $H$ is normal in $G$.

[2] Classify the abelian groups of cardinality $10^5$. Explain how/why you know you have all of them.

[3] Show that two linear operators $S, T$ on a finite-dimensional complex vectorspace, with $ST = TS$, have at least one (non-zero) simultaneous eigenvector.

[4] Show that the ideal in $\mathbb{Z}[x]$ generated by $x^2 + 1$ and $19$ is maximal.

[5] Show that $x^4 + 1$ is reducible in $\mathbb{F}_p[x]$ for every prime $p$.

[6] Let $\omega$ be a primitive fifth root of unity in $\mathbb{C}$. Show that $\sqrt{5} \in \mathbb{Q}(\omega)$.

[7] Prove that there is a unique isomorphism class of $\mathbb{Z}[i]$-module of cardinality $7^2$.

[8] Give an example of a short exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ of abelian groups, and an abelian group $X$, such that

$$0 \rightarrow \text{Hom}(X, A) \rightarrow \text{Hom}(X, B) \rightarrow \text{Hom}(X, C) \rightarrow 0$$

is not exact.