Complex Analysis Prelim Written Exam *Spring 2017: discussion*

[1] Write three terms of the Laurent expansion of \( f(z) = \frac{e^z - 1}{z(z - 1)} \) centered at 0 and convergent in \(|z| > 1\).
**Discussion:** Unlike some previous exams, this is not pure algebra.

[2] Show that an \( \mathbb{R} \)-valued holomorphic function \( f \) is constant.
**Discussion:** Iconic.

[3] Evaluate \( \int_{-\infty}^{\infty} \frac{e^{ix}}{1 + x^2} \, dx \).
**Discussion:** Iconic.

[4] Determine the radius of convergence of the power series for \( \frac{z^2}{1 - \cos z} \) at 0.
**Discussion:** [... iou ...]

[5] Let \( f \) be an entire function such that \(|f(z)| \leq 1 + \sqrt{|z|} \) for all \( z \in \mathbb{C} \). Show \( f \) is constant.
**Discussion:** Iconic.

[6] Show that there is a holomorphic function \( f \) on the region \(|z| > 2\) such that \( f(z)^4 = z^4 + z + 1 \).
**Discussion:** Nearly iconic. [... iou ...]

[7] Show that \( \frac{\pi^2}{\sin^2 \pi z} = \sum_{n \in \mathbb{Z}} \frac{1}{(z - n)^2} \)
**Discussion:** Iconic, ... with some pitfalls...

[8] Make a change of coordinates to put the elliptic curve \( w^2 = z^4 + 1 \) into the (essentially) Weierstraß form \( y^2 = x^3 + bx + c \).
**Discussion:** This is a 200-year-old thing, but perhaps not so well-known. Also, it has some importance, since a mild extension of it shows that every elliptic curve over \( \mathbb{C} \) can be put into the form \( y^2 = x(x - 1)(x - \lambda) \), with a unique \( \lambda \in \mathbb{C} \).

Ok, so how to do it? Send one of the zeros of \( z^4 + 1 \) to infinity, by a linear fractional transformation, e.g., \( w = \frac{1}{z - \zeta_8} \), with \( \zeta_8 \) a primitive eighth root of unity. Then express \( z \) in terms of \( w \), make a change of variables in \( y \), and you’ll have a cubic. More later [... iou ...].