Complex Analysis Prelim Written Exam Autumn 2021

Questions are equally weighted. Give essential explanations and justifications: a large part of each question is demonstration that you understand the context and understand which issues are primary. Do not choose assumptions or contexts making the problems silly. Coherent writing is essential: your paper should not be a puzzle for the grader.

Please follow the rules for this exam, as explained by the proctor and the office of the Director of Graduate Studies in Mathematics.

[1] Write the Laurent series expansion for \( f(z) = \frac{1}{z^2 - 1} \) centered at \( z = 0 \) and converging in \(|z| > 1\).

[2] Give a conformal mapping from the quadrant \( \{z : \Re(z) > 0, \Im(z) > 0\} \) to the disk \( \{z : |z| < 1\} \).

[3] Determine the radius of convergence of the power series at \( z = 0 \) for \( \sqrt{\frac{\sin z}{z}} \).

[4] Classify entire functions \( f \) such that \( |f(z)| \leq \sqrt{|z|} \) for all \( z \in \mathbb{C} \).

[5] Evaluate \( \int_0^\infty \frac{\cos t}{1 + t^2} \, dt \).

[6] Let \( f \) be a holomorphic function on \( \{z : |z| < 2\} \). Show that for all sufficiently small \( \varepsilon > 0 \) the function \( z^n + \varepsilon f(z) \) has exactly \( n \) zeros inside \( |z| = 1 \).

[7] Exhibit a (not-identically-zero) harmonic function on the punctured disk \( \{z : 0 < |z| < 1\} \), extending continuously to \( \{z : 0 < |z| \leq 1\} \), which is identically 0 on the circle \( \{z : |z| = 1\} \).

[8] Show that the curve \( z^2 = w^4 + 1 \) is an elliptic curve.