Complex Analysis Prelim Written Exam Spring 2022

Questions are equally weighted. Give essential explanations and justifications: a large part of each question is demonstration that you understand the context and understand which issues are primary. Do not choose assumptions or contexts making the problems silly. Coherent writing is essential: your paper should not be a puzzle for the grader.

Please follow the special rules for this exam, as explained by the proctor and the office of the Director of Graduate Studies in Mathematics.

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[1] Determine the radius of convergence of the power series for $\sqrt{z}$ at $z_o = -3 + 4i$.

[2] Write three non-zero terms of the Laurent expansion of $f(z) = \frac{1}{z(z-1)(z-2)}$, centered at 0, and convergent in $1 < |z| < 2$.

[3] Evaluate $\int_0^\infty \frac{e^{itx}}{1 + x^2} \, dx$ for real $t$.

[4] Let $f$ be a holomorphic function on $\{z : |z| < 2\}$. Show that for all sufficiently small $\varepsilon > 0$ the function $z^n + \varepsilon f(z)$ has exactly $n$ zeros inside $|z| = 1$.

[5] For non-vanishing holomorphic $f$, show that $\log |f(z)|$ is harmonic.

[6] Find a non-identically-zero harmonic function $u$ on the open unit disk whose radial limits (exist and) are 0 at every point of the unit circle except at 1.

[7] Change variables on the elliptic curve $z^2 = w^4 - 1$ to write it in the approximately Weierstraß form $y^2 = \text{cubic in } t$. (Hint: move one of the zeros of the quartic to $\infty$.)