Real Analysis Prelim Written Exam  Fall 2016

Questions are equally weighted. Give essential explanations and justifications: a large part of each question is demonstration of understanding of the context and of which issues are primary. Do not make assumptions or choose contexts making the problems silly. Coherent writing is essential: your paper should not be a puzzle for the grader.

Write your codename, not actual name, on each booklet. No notes, books, calculators, computers, cell phones, wireless, bluetooth, or other communication devices may be used during the exam.

[1] Given $\varepsilon > 0$, construct an open set $U \subset \mathbb{R}$ containing $\mathbb{Q}$ and with Lebesgue measure less than $\varepsilon$.

[2] Give an example of a complete metric space that is not locally compact.

[3] Use Parseval’s identity to show that $\sum_{n \in \mathbb{Z}} \frac{1}{n^2} = \frac{\pi^2}{6}$.

[4] Give an example of a sequence $\{f_n\}$ of continuous functions on $[0, 1]$ such that $f(x) = \lim_n f_n(x)$ is continuous, $\int_0^1 f_n(t) \, dt = 1$ for all $n$, but $\int_0^1 f(t) \, dt = 0$.

[5] Give an example of a continuous function $f$ on $[0, 1]$ which is differentiable almost everywhere, with Lebesgue-integrable derivative $f'$, but $\int_0^1 f'(t) \, dt \neq f(1) - f(0)$.

[6] Let $f \in L^2[0, 1]$ be absolutely continuous and have derivative $f' \in L^2[0, 1]$. Show that there is a constant $C$ such that $|f(x) - f(y)| < C \cdot |x - y|^\frac{3}{2}$ for $x, y \in [0, 1]$.

[7] Suppose that $f \in L^1(\mathbb{R})$ and $\int_a^b f(x) \, dx = 0$ for all $a, b \in \mathbb{Q}$. Show that $f(x) = 0$ almost everywhere.

[8] Let $E$ be a Lebesgue measurable subset of $[0, 1]$, and let $f(x) = \int_E \sin(tx) \, dt$. Show that $f(x)$ is continuous.