Real Analysis Prelim Written Exam Fall 2017

Questions are equally weighted. Give essential explanations and justifications: a large part of each question is demonstration of understanding of the context and of which issues are primary. Do not make assumptions or choose contexts making the problems silly. Coherent writing is essential: your paper should not be a puzzle for the grader.

Write your codename, not actual name, on each booklet. No notes, books, calculators, computers, cell phones, wireless, bluetooth, or other communication devices may be used during the exam.

[1] Exhibit a sequence \( \{f_n\} \) of continuous functions on \([0, 1]\) such that \( \lim_{n} f_n(x) = 0 \) for all \( x \in [0, 1] \), but \( \int_0^1 f_n(t) \, dt = 1 \) for all \( n \).

[2] Show that, for \( f \in L^1(\mathbb{R}) \), for \( \varepsilon > 0 \), there is \( \delta > 0 \) such that \( \int_{|x| \leq \delta} |f(x)| \, dx < \varepsilon \).

[3] Let \( E \subset \mathbb{R} \) be Lebesgue measurable with finite measure, and \( f(x) = \int_E \sin(tx) \, dt \). Show that \( f(x) \) goes to 0 at infinity.

[4] Let \( C = \{v = (v_1, v_2, \ldots) \in \ell^2 : |v_n| \leq \frac{1}{n}\} \subset \ell^2 \). Show that \( C \) is compact.

[5] Compute \( \int_{\mathbb{R}} \sin^2 x \, dx \).

[6] Show that \( C^1[a, b] \) with norm \( |f|_{C^1} = \sup_{x \in [a, b]} |f(x)| + \sup_{x \in [a, b]} |f'(x)| \) is a Banach space.

[7] Let \( f \in L^2[0, 1] \), with distributional derivative \( f' \in L^2[0, 1] \). Show that there is a constant \( C \) such that \( |f(x) - f(y)| < C \cdot |x - y|^{\frac{1}{2}} \) for \( x, y \in [0, 1] \). (Note: evidently the hypotheses are not satisfied by the Cantor-Lebesgue staircase function.)