[1] Give an example of $f \in L^1(\mathbb{R})$ with $\text{ess sup}_{x \to +\infty} |f(x)| = +\infty$. That is, ignoring irrelevant tricks on sets of measure 0, $|f(x)|$ is unbounded as $x \to +\infty$.

[2] For $f \in L^1(\mathbb{R})$, show that, for every $\varepsilon > 0$, $\lim_{n \to +\infty} \int_{n}^{n+\varepsilon} |f(x)| \, dx = 0$, as $n$ runs through positive integers.

[3] For a non-compact Lebesgue-measurable subset $E$ of $\mathbb{R}$ with finite Lebesgue measure, show that $f(x) = \int_{E} \cos(t) \sin(tx) \, dt$ is a continuous function of $x$.

[4] Show that the closed unit ball in $L^2[0,1]$, while closed and bounded, is not compact.

[5] Show that there is no function $h \in C^0[0,1]$ such that

$$\int_{0}^{1} h(x) f(x) \, dx = f(\frac{1}{2}) \quad \text{(for all } f \in C^0[0,1])$$

[6] Show that $C^1[a,b]$ with norm $|f| = \sup_{x \in [a,b]} |f(x)|$ is not complete.

[7] Let $f \in L^2[0,1]$, with distributional derivative $f' \in L^2[0,1]$. Show that $f$ is continuous: in particular, show that there is a constant $C$ such that

$$|f(x) - f(y)| < C \cdot |x - y|^{\frac{1}{2}} \quad \text{(for } x, y \in [0,1])$$

[8] Show that the Volterra operator $Vf(x) = \int_{0}^{x} f(t) \, dt$ has no eigenvectors, and has spectrum $\{0\}$. 