# Fourth-Quarter Calculus Exercises



see also http://www.math.umn.edu/ garrett/calculus/3251/

---

### 11.1 coordinates, analytic geometry

**#1.** Given two points $A = (1, 2), B = (2, 4)$, suppose these are opposite vertices of a rectangle which has sides parallel to the coordinate axes. Find the coordinates of the *other* two vertices of this rectangle.

**#2.** Given two points $A = (0, 0, 0), B = (2, 3, 4)$, suppose these are opposite vertices of a rectangular box which has sides parallel to the coordinate axes. Find the coordinates of the *other* six vertices of this box.

**#3.** Given two points $A = (-1, 0, 1), B = (2, 3, 4)$, suppose these are opposite vertices of a rectangular box which has sides parallel to the coordinate axes. Find the coordinates of the *other* six vertices of this box.

**#4.** Remember that a triangle is **scalene** if all its sides are of different lengths, is **isosceles** if *at least* two sides are the same lenght, and *equilateral* if all sides are the same length. A triangle is **right** (?!) if one of the angles in it is $90^o$. Consider a triangle with vertices $A = (5, 5, 1), B = (3, 3, 2)$, and $C = (1, 4, 4)$, and tell which of these adjectives, if any, apply.

**#5.** Tell whether the points $A = (0, 0, 0), B = (1, 2, 3)$, and $C = (2, 4, 6)$ are *collinear* or not.

**#6.** Tell whether the points $A = (0, 0, 0), B = (1, 2, -2)$, and $C = (3, 0, 1)$ are *collinear* or not.

**#7.** Tell whether the points $A = (1, 1, 1), B = (1, 2, 3)$, and $C = (1, 3, 5)$ are *collinear* or not.

**#8.** Write the equation of the sphere with center $(-1, 2, 4)$ and radius $\frac{1}{2}$.

**#9.** Rewrite the equation

$$x^2 + y^2 + z^2 + ax + by + cz + d = 0$$

in such a manner that you can tell the radius and center of the sphere described by the equation. (We suppose that $a^2 + b^2 + c^2 > 4d$).

**#10.** Describe the set of points which are equi-distant from the points $(1, 2, 3)$ and $(1, 1, 1)$. What is this set of points, in geometric terms?

**#11.** Describe in more geometric terms the region *in three-space* described by $x^2 + y^2 \leq 1$.

**#12.** Describe in more geometric terms the region in three-space described by $16 \leq x^2 + y^2 + z^2 \leq 25$.

---

### 11.2 vectors

**#13.** Find the sum of the two vectors $(1, 2, 3)$ and $(5, 4, 2)$.

**#14.** Let $\vec{v} = (1, 2, 2)$ and $\vec{w} = (2, 3, 4)$. Compute

$$|\vec{v}|, \;\; \vec{v} + \vec{w}, \;\; 2\vec{v} - \vec{w}, \;\; 3\vec{v}$$

**#15.** Let $\vec{v} = (0, 5, 12)$ and $\vec{w} = (1, 0, 1)$. Compute

$$|\vec{v}|, \;\; \vec{v} + \vec{w}, \;\; 2\vec{v} - \vec{w}, \;\; 3\vec{v}$$

**#16.** Let $\vec{v} = (3, 0, 4)$ and $\vec{w} = (1, 3, 0)$. Compute

$$|\vec{v}|, \;\; \vec{v} + \vec{w}, \;\; 2\vec{v} - \vec{w}, \;\; 3\vec{v}$$

**#17.** Let $\vec{v} = 0\vec{\imath} + 7\vec{\jmath} + 24\vec{k}$ and $\vec{w} = 1\vec{\imath} + 0\vec{\jmath} + 1\vec{k}$. Compute

$$|\vec{v}|, \quad \vec{v} + \vec{w}, \quad 2\vec{v} - \vec{w}, \quad 3\vec{v}$$

**#18.** Find a unit vector in the direction of $(0, 3, 4)$.

**#19.** Find a unit vector in the direction of $(1, 2, 2)$.

**#20.** Find a unit vector in the direction of $2\vec{\imath} + 3\vec{\jmath} + 6\vec{k}$.

**#21.** By solving a little system of two linear equations in two unknowns, express $\vec{\imath}$ and $\vec{\jmath}$ as *linear combinations* of $(1, 2)$ and $(2, 5)$.

**#22.** By solving a little system of two linear equations in two unknowns, express $(2, 3)$ and $(3, 4)$ as *linear combinations* of $(1, 2)$ and $(2, 5)$.

**#23.** Suppose that a runway runs from Southwest to Northeast. Suppose that there is a 50*mph crosswind* blowing from the Southeast. Suppose that a pilot must land on that runway with groundspeed 120*mph*. What airspeed and what 'heading' must the pilot have to achieve this?

---

### 11.3 Dot product, angles

**#24.** Compute the dot product of $(1, 2, 3)$ and $(2, 3, 4)$.

**#25.** Compute the dot product of $\vec{\imath} + \vec{\jmath}$ and $2\vec{\imath} + \vec{\jmath} - \vec{k}$.

**#26.** Compute the cosine of the angle between $(1, 2, 3)$ and $(2, 3, 4)$.

**#27.** Compute the cosine of the angle between $\vec{\imath} + \vec{\jmath}$ and $2\vec{\imath} + \vec{\jmath} - \vec{k}$.

**#28.** Are the vectors $(2, 2, 1), (2, 3, -10)$ *orthogonal*?

**#29.** Are the vectors $\vec{\imath} + \vec{\jmath} + \vec{k}$ $\vec{\imath} - 3\vec{\jmath} + \vec{k}$*orthogonal*?

**#30.** Are the vectors $(2, 2, 1), (2, 3, -10)$ *parallel*?

**#31.** Are the vectors $\vec{\imath} + \vec{\jmath} + \vec{k}$ $2\vec{\imath} + 2\vec{\jmath} + \vec{k}$*parallel*?

**#32.** For what values of the scalar $t$ are the two vectors $(1, 3, 2), (t^2, t, 1)$ *orthogonal*?

**#33.** For what values of the scalar $t$ are the two vectors $\vec{\imath} + \vec{\jmath} + \vec{k}, \vec{\imath} + t^2\vec{\jmath} - 2\vec{k}$ *orthogonal*?

**#34.** Find the so-called *direction cosines* and *direction angles* of the vectors $(1, 2, 3)$, $\vec{\imath} + \vec{\jmath} + \vec{k}$.

**#35.** The *long diagonals* of a cube are the lines connecting *opposite* vertices of the cube. (These pass through the center of the cube). Find the cosine of the angle between two such long diagonals.

---

### 11.4 Cross product

**#36.** Find $(1, 2, 3) \times (2, 3, 4)$.

**#37.** Find $(\vec{\imath} - \vec{\jmath} + \vec{k}) \times (2\vec{\imath} + \vec{\jmath} + 2\vec{k})$.

**#38.** Find a (non-zero) vector orthogonal to both $(1, 2, 3)$ and $(1, 1, 1)$.

**#39.** Find a vector (non-zero) orthogonal to both $\vec{\imath} + 2\vec{\jmath} + 5\vec{k}$ and $2\vec{\imath} - \vec{\jmath} + \vec{k}$.

**#40.** Find a vector orthogonal to the plane through the points $(0, 0, 0)$, $(1, 3, 2)$, $(3, 2, 1)$.

**#41.** Find two unit vectors, both orthogonal to both $(1,2,3)$ and $(1,0,1)$.

**#42.** Find the area of a parallelogram with vertices $(0,0,0)$, $(1,2,3)$, $(1,1,1)$, $(2,3,4)$.

**#43.** Find the area of a parallelogram with vertices $(1,1,1)$, $(1,2,3)$, $(2,3,4)$, $(2,4,6)$.

**#44.** Find the volume of the parallelopiped determined by vectors $(1,1,1)$, $(1,2,3)$, $(2,2,1)$.

**#45.** Verify that the points $(0,0,0)$, $(1,1,1)$, $(1,2,3)$, $(4,6,8)$ are *coplanar*.

---

### 11.5 Lines, Planes

**#46.** Write a vector equation, a parametrized form, and a 'symmetric equation' for the line through the point $(1,2,3)$ in direction $(2,1,5)$.

**#47.** Write a vector equation, a parametrized form, and a 'symmetric equation' for the line through the points $(1,2,3)$ and $(2,1,2)$.

**#48.** Show that the line through $(1,1,1)$ and $(8,2,3)$ is perpendicular to the line through $(1,1,1)$ and $(2,-2,-1)$.

**#49.** Find a line through $(1,1,1)$ and perpendicular to the plane $x-y+2z=2$.

**#50.** Tell whether the following two lines are *parallel*, *skew*, or they *intersect*, and if they intersect tell the intersection point:

$$\frac{x-1}{1}=\frac{y-2}{1}=\frac{z+1}{2}$$

and

$$\frac{x-1}{2}=\frac{y+2}{2}=\frac{z-1}{1}$$

**#51.** Find the plane through $(1,2,3)$ with normal vector $(1,1,1)$.

**#52.** Find the plane through $(1,1,1),(1,2,3),(2,1,2)$.

**#53.** Find the plane through $(1,1,1)$ and also containing the line

$$\frac{x-1}{1}=\frac{y+1}{2}=\frac{z-2}{2}$$

**#54.** Find the intersection of the plane $x+y+2z=6$ and the line given in parametrized form by $x=3-t, y=2-t, z=3t-2$.

**#55.** Find the cosine of the angle between the planes $x+y+2z=6$ and $x-y-3z=2$.

**#56.** Find the cosine of the angle between the plane $x+y+z=2$ and the line $x=t+1, y=t-1, z=3t$.

**#57.** Determine the line which is the intersection of the two planes $x+y+z=2$ and $2x-y+z=6$.

**#58.** Find the distance from the point $(1,2,3)$ to the plane $x+y-2z=6$.

---

### Quadric surfaces 11.6

**#59.** Put the following equation in *normal form* and tell what kind of surface it is: $x^2+z^2=y^2-1$.

**#60.** Put the following equation in *normal form* and tell what kind of surface it is: $x^2+z^2=y-1$.

**#61.** Put the following equation in *normal form* and tell what kind of surface it is: $x^2+y=2x-z^2$

**#62.** Put the following equation in *normal form* and tell what kind of surface it is: $x^2 - 1 = y^2 + z^2$

**#63.** Put the following equation in *normal form* and tell what kind of surface it is: $x^2+y^2 = z-2y-2x+1$

---

### vector functions, curves, tangents 11.7

**#64.** Find the derivative of $\vec{f}(t) = (t, t^2, t^3)$.

**#65.** Find the derivative of $\vec{f}(t) = (t, t^2, t^3) \times (e^t, e^{-t}, 0)$.

**#66.** Find the derivative of $\vec{f}(t) = (\arctan t, \ln(t^2-1), \sqrt{1-t^3})$. For what values of $t$ can we make sense of this function, that is, what is the largest possible *domain* it could have?

**#67.** Compute

$$\frac{d}{dt}[(t\vec{a} + \vec{b}) \times (t\vec{a} - \vec{b})]$$

where $\vec{a}$ and $\vec{b}$ are constant vectors.

**#68.** Find the derivative of $\vec{f}(t) = (t, t^2, t^3) \cdot (e^t, e^{-t}, 0) \times (1, t, 1)$.

**#69.** Find the derivative of $\vec{f}(t) = (t, t^2, t^3) \cdot [(1, 2, t) - (e^t, \cos 3t, \sin 3t)]$.

**#70.** Write a formula for the *unit* tangent vector $\vec{T}(t)$ to the parametrized curve $\vec{f}(t) = (\cos 3t, \sin 3t, 4)$. Write an equation for the tangent line at a point $\vec{x_o} = \vec{f}(t_o)$.

**#71.** Find the point where the curves $\vec{f}(t) = (t, t^2, t^3)$ and $\vec{g}(t) = (2t-1, 1-2t, t^2-1)$ intersect, and determine the angle of intersection.

---

### arc length and curvature 11.8

**#72.** Find the length of the **helix**

$$\vec{h}(t) = (r\ \cos ct, r\ \sin ct, kt)$$

from $t = a$ to $t = b$.

**#73.** Find the length of

$$\vec{f}(t) = (e^t, \cos t\, e^t, \sin t\, e^t)$$

from $t = a$ to $t = b$.

**#74.** Find the length of

$$\vec{f}(t) = (2t, t^2, \frac{t^3}{3})$$

from $t = 0$ to $t = 3$.

**#75.** Find the *curvature* of the plane curve $y = \sqrt{1-x^2}$.

**#76.** Find the *curvature* of the plane curve $y = x^2$.

**#77.** Find the *curvature* of the plane curve $y = \ln x$.

**#78.** Find the *curvature* of the parametrized plane curve $(x, y) = (r\ \cos ct, r \sin ct)$.

**#79.** Find the *curvature* of the parametrized plane curve $(x, y) = (t^3, t^2)$.

**#80.** Find the *unit tangent vector*, the *principal unit normal* vector, and the *curvature* of the **helix**

$$\vec{h}(t) = (r\ \cos ct, r\ \sin ct, kt)$$

**#81.** Find the *unit tangent vector*, the *principal unit normal* vector, and the *curvature* of $\vec{f}(t) = (t, t^2, t^3)$.

---

### velocity and acceleration 11.9

**#82.** Find the tangential and normal components of the acceleration for a particle moving along a *helical* path, with *position* given by $\vec{f}(t) = (5\cos t, 5\sin t, 12t)$.

**#83.** If a projectile is shot at a $45^o$ angle upward at $64\sqrt{2}$ feet per second from a 192-foot-tall tower, how far away will it land?

**#84.** (This problem is much more serious than the others here, but is also more genuine!)In the situation of the previous example, what angle *maximizes* the horizontal distance travelled by the projectile?

**#85.** In both the previous situations: How long will the projectile be in the air? What is the maximal altitude reached by the projectile? What is the speed at impact?

---

### Cylindrical, Spherical coordinates 11.10

**#86.** Rewrite the point $(x, y, z) = (\sqrt{2}, \sqrt{2}, \sqrt{5})$ (given in rectangular coordinates) in cylindrical coordinates $(r, \theta, z)$ and in spherical coordinates $(\rho, \theta, \varphi)$.

**#87.** Rewrite the point $(x, y, z) = (1, 0, 0)$ (given in rectangular coordinates) in cylindrical coordinates $(r, \theta, z)$ and in spherical coordinates $(\rho, \theta, \varphi)$.

**#88.** Rewrite the point $(r, \theta, z) = (1, \frac{\pi}{2}, 0)$ (given in cylindrical coordinates) in rectangular coordinates $(x, y, z)$ and in spherical coordinates $(\rho, \theta, \varphi)$.

**#89.** Rewrite the point $(\rho, \theta, \varphi) = (1, \pi, \frac{\pi}{2})$ (given in spherical coordinates) in rectangular coordinates $(x, y, z)$ and in cylindrical coordinates $(r, \theta, \varphi)$.

**#90.** Rewrite in spherical coordinates $(\rho, \theta, \varphi)$ the surface described in rectangular coordinates by $x^2 + y^2 + z^2 = 36$.

**#91.** Rewrite in cylindrical coordinates the parametrized curve *(helix!)* $\vec{f}(t) = (3\cos t, 3\sin t, t)$.

**#92.** Rewrite in rectangular coordinates the surface which in spherical coordinates is given by $\varphi = \frac{\pi}{2}$.

---

### Functions of several variables 12.1

**#93.** Describe the level curves of the function $f(x, y) = x^2 - y^2 + 2x$.

**#94.** Describe the level surfaces of the function $f(x, y, z) = x^2 + y^2 + z^2$.

**#95.** Describe the level surfaces of the function $f(x, y, z) = x^2 + y^2 - z^2$.

---

### Limits and continuity 12.2

**#96.** Find $\lim_{(x,y)\to(1,2)} xy - x^2 + y^3$.

**#97.** Find $\lim_{(x,y)\to(1,2)} \frac{xy-x^2+y^3}{x^2+y^2}$.

**#98.** Find $\lim_{(x,y)\to(0,0)} \frac{x^4-y^4}{x^2+y^2}$ by using a *cancellation trick*

**#99.** Show that $\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2} = 0$, by *sandwiching* the $x^3$ between $-(x^2+y^2)^{3/2}$ and $(x^2+y^2)^{3/2}$ and invoking what is sometimes called a **Sandwich Lemma**.

**#100.** Show that $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ *does not exist*, by approaching $(0,0)$ along different straight lines $y = cx$ and seeing that the limits *along these lines* are all different, depending upon $c$.

---

**Partial derivatives 12.3**

**#101.** Compute

$$\frac{\partial}{\partial x}(3x^2 - xy + e^{xy})$$

**#102.** Compute

$$\frac{\partial}{\partial y}(3x^2 - xy + e^{xy})$$

**#103.** Compute

$$\frac{\partial}{\partial x}(\frac{x^2y^2 - \cos(x+y)}{x^2+y^2})$$

**#104.** Compute

$$\frac{\partial}{\partial z}(e^{xyz} - \sqrt{xy+yz+zx})$$

**#105.** Compute

$$\frac{\partial^2}{\partial z\,\partial y}(e^{xyz} + xy + yz + zx)$$

**#106.** Compute

$$\frac{\partial^2}{(\partial z)^2}(e^{xyz} + xy + yz + zx)$$

---

**Gradients, directional derivatives 12.4 and 12.6**

**#107.** Find the gradient $\nabla f$ of $f(x,y,z) = x^2 + y^2 - z^2$.

**#108.** Find the directional derivative of $f(x,y,z) = x^2 + y^2 - z^2$ at $(1,0,0)$ in the direction $(3,4,0)$.

**#109.** Find the directional derivative of $f(x,y,z) = x^2 + y^2 - z^2$ at $(1,2,3)$ in the direction $(0,1,0)$.

**#110.** Find the direction of greatest increase of $f(x,y) = x^2 + xy + y^2$ at the point $(1,2)$. What is the rate of change in that direction?

---

**Normals to level surfaces, tangent planes 12.4 and 12.6 again**

**#111.** Find a vector perpendicular to the surface $z = x^2 + y^2$ at the point $(1,2,5)$.

**#112.** Find an equation of the tangent plane to the sphere $x^2 + y^2 + z^2 = 14$ at the point $(1,2,3)$.

**#113.** Find a line perpendicular to the surface $x = y^2 - z^2$ and passing through the point $(1,1,1)$.

---

**Approximations, round-off error 12.4**

**#114.** Use 'approximation by differentials' to estimate $\sqrt{x+1}$ in terms of $\sqrt{x}$.

**#115.** Two numbers between 0 and 10, are rounded off to the nearest $1/100^{th}$, and then multiplied. Use 'differentials' to estimate the error in the product. That is, estimate the difference between the actual product and the product of the rounded-off numbers.

**#116.** Five numbers between 0 and 10, are rounded off to the nearest $1/100^{th}$, and then multiplied. Use 'differentials' to estimate the error in the product. That is, estimate the difference between the actual product and the product of the rounded-off numbers.

**#117.** A number $x$ between 0 and 5 is rounded off to the nearest $1/1000^{th}$. Let $x_o$ be the rounded-off version of $x$. Use differentials to estimate the difference between $e^x$ and $e^{x_o}$.

---

### Chain Rule 12.5

**#118.** Let $f(x, y, z) = xy + yz$ and let $F(s, t) = f(s, st, t^2)$. What is $\partial F/\partial t$?

**#119.** Let $f(x, y, z) = xyz$ and let $F(s, t) = f(t, s, s + t)$. What is $\partial F/\partial s$?

**#120.** Let $f(x, y, z) = xyz$ and let $x = u(s, t), y = v(s, t), z = w(s, t)$, where $u(s, t) = s + t, v(s, t) = s - t$, and $w(s, t) = t$. Find the partial derivative of $f(u, v, w)$ with respect to $t$.

**#121.** Let $f(x, y, z) = x + y + z$ and let $F(x, y) = f(x, y, xy)$. What is $\partial F/\partial x$?

**#122.** Let $f(x, y, z) = xyz$ and let $u(s, t) = s+t, v(s, t) = s-t$, and $w(s, t) = t$. Find the partial derivative of $f(u(x, y), v(x, y), w(x, y))$ with respect to $y$.

**#123.** Let $g(x, y) = x^2 - y^2$, let $u(x, y) = x - y, v(x, y) = x + y$, and let $a(t) = t^3$, $b(t) = e^t$. Define

$$F(t) = g(u(a(t), b(t)),\ v(b(t), a(t)))$$

Find $dF/dt$.

**#124.** Let $f, g$ be two nice functions, and define $F(x, t) = f(x - t) + g(x + t)$. Show that $F$ satisfies what is called the **one-dimensional wave equation**, meaning that

$$\frac{\partial^2 F}{\partial x^2} - \frac{\partial^2 F}{\partial t^2} = 0$$

---

### Implicit Differentiation 12.5

**#125.** Define $y$ as a function of $x$ by the relation $y^3 - xy + x^3 = 1$. Express $dy/dx$ as a function of $x$ *and also* $y$. Express $d^2y/dx^2$ as a function of $x, y$, and of $dy/dx$. Find $dy/dx$ at the point where $x = y = 1$.

**#126.** Define $z$ as a function of $x$ and $y$ by the relation $z^3 + xyz + x^3 + y^3 = 4$. Express $\partial z/\partial y$ as a function of $x, y, z$.

**#127.** Define $y$ as a function of $x$ and $z$ by the relation $z^3 + xyz + x^3 + y^3 = 4$. Express $\partial y/\partial z$ as a function of $x, y, z$.

**#128.** Define $x$ as a function of $y$ and $z$ by the relation $e^x + e^y + e^z + xyz = 3$. Find $\partial x/\partial z$ at the point $(0, 0, 0)$.

**#129.** Define $x$ as a function of $y, z$ by requiring that

$$F(x, u(x, y), v(z, x, y)) = 0$$

for some functions $F, u, v$. What is $\partial x/\partial z$?

---

## Critical points, saddle points 12.7

**#130.** Find the critical points of $f(x,y) = x^2 + y^2$ and identify the relative maxima, relative minima, and saddle points among them.

**#131.** Find the critical points of $f(x,y) = x^2 - y^2$ and identify the relative maxima, relative minima, and saddle points among them.

**#132.** Find the critical points of $f(x,y) = x^2 + 2x + y^2 - 6y$ and identify the relative maxima, relative minima, and saddle points among them.

**#133.** Find the critical points of $f(x,y) = x^2 + xy + y^2$ and identify the relative maxima, relative minima, and saddle points among them.

**#134.** Find the critical points of $f(x,y) = x^2 + 3xy + y^2$ and identify the relative maxima, relative minima, and saddle points among them.

**#135.** Find the critical points of $f(x,y) = x^3 - 3xy + y^3$ and identify the relative maxima, relative minima, and saddle points among them.

**#136.** Find the critical points of $f(x,y) = 2x^3 - 3x^2y + y^3$ and identify the relative maxima, relative minima, and saddle points among them.

**#137.** Find the critical points of $f(x,y) = (e^x + e^{-x})(e^y + e^{-y})$ and identify the relative maxima, relative minima, and saddle points among them.

---

## Lagrange multipliers 12.8

**#138.** *Using Lagrange multipliers*, find the points on $x^2 + y^2 = 25$ where the minimum and maximum of the function $3x - 4y$ occur, and the *values* of them.

**#139.** *Using Lagrange multipliers*, find the points on $x^2 + y^2 = 25$ where the minimum and maximum of the function $xy$ occur, and find the minimum and maximum *values*.

**#140.** *Using Lagrange multipliers*, find the points on $x^2 + y^2 = 25$ where the minimum and maximum of the function $x^2 - y^2$ occur, and find the minimum and maximum *values*.

**#141.** *Using Lagrange multipliers*, find the points on $x^4 + y^4 = 17$ where the minimum and maximum of the function $x + 8y$ occur, and find the minimum and maximum *values*.

**#142.** *Using Lagrange multipliers*, find the points on $x^4 + y^4 = 1$ where the minimum and maximum of the function $x^2 + 2xy + y^2$ occur, and find the minimum and maximum *values*.

**#143.** Find the maxima and minima of $x^2 + 4xy + y^2$ on $x^2 + y^2 \leq 2$, and tell where they occur.

**#144.** Find the maxima and minima of $x^2 + 4xy + y^2 - 4x + 4y$ on $x^2 + y^2 \leq 1$, and tell where they occur.