Notes on fourth-quarter calculus

see also http://www.math.umn.edu/ garrett/calculus/3251/

1. Coordinates:

in two- and three-space, right-hand rule, 'locus' as old-fashioned terminology for 'set of points'. Distance formula (from the Pythagorean Theorem):

$$\text{distance from } (x_1, y_1, z_1) \text{ to } (x_2, y_2, z_2) =$$

$$= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

This is the fundamental expression of geometry in terms of algebra, since nearly all issues about Euclidean geometry can be expressed as issues about *distance*.

2. Vector Algebra in two- and three-space:

Without worrying about what 'higher dimensions *are*', for all purposes of computation or of *proving* things theoretically,

$$\text{vector in two-space} = \text{ordered pair of numbers}$$

$$\text{vector in three-space} = \text{ordered triple of numbers}$$

$$\text{vector in four-space} = \text{ordered quadruple of numbers}$$

and so on.

Usually letters to denote vectors have *arrows* on them to *emphasize* that they are vectors and not just numbers, as in $\vec{x}$ rather than plain x. *But this is not really required.*

In the context of discussion of *vectors*, for some reason plain *numbers* are called **scalars**.

$\mathbf{R}^2$ is standard notation for two-dimensional space, $\mathbf{R}^3$ is standard notation for three-dimensional space, and generally $\mathbf{R}^n$ is standard notation for n-dimensional space.

The *interpretations* of vectors as *positions* ('dots') or *directions* ('arrows') are just that, *interpretations.* The question of whether some vector *is* a *position* or a *direction* depends entirely upon the *context*, and therefore upon the *purpose* to which it is being used. *Vectors in themselves are neither 'dots' nor 'arrows', and are neither positions nor directions.*

There are some standard abbreviations for important vectors:

$$\vec{\imath} = (1,0) \quad \vec{\jmath} = (0,1) \qquad \text{(in two-space)}$$

$$\vec{\imath} = (1,0,0) \quad \vec{\jmath} = (0,1,0) \quad \vec{k} = (0,0,1) \qquad \text{(in three-space)}$$

Note that again *context* is critical: for example, the notation '$\vec{\imath}$' may be *either* $(1,0)$ or $(1,0,0)$, depending entirely on context.

Concepts: vector addition, scalars (number), scalar multiplication, zero-vector, linear combinations, linear independence/dependence, 'resultant' of two vectors is archaicism for 'sum', velocity vectors versus magnitudes, called *speed*, *magnitude and direction* as in polar coordinates.

Properties of vector algebra operations such as distributive and associative laws.

3. Dot products:

=scalar products = inner products: this is *vector geometry*.

length (norm), angles, orthogonal=perpendicular=normal, unit vector, *Amazing Formula* for cosine of angle, direction cosines (=angles with the three axes)

Algebraic properties of dot products in combination with other vector algebra operations.

4. Cross products:

Review of two-by-two and three-by-three determinants

Algebraic properties, finding vectors orthogonal to two given ones, volumes of parallelopipeds, areas of parallelograms, *torque* (definition)

5. Lines and planes:

With techniques of vector algebra, many elementary *computational* questions about lines and planes can be easily answered. The point is that, even when our intuition can give us an idea about *qualitative* behavior of lines and planes, issues about *quantitative* behavior can be very hard to answer simply *without vector algebra and geometry.*

'vector parametric equation' versus 'scalar parametric equation*s*' versus 'symmetric equations' of lines (in any number of dimensions),

'direction numbers' of planes, (=) orthogonal ('normal') vector to planes, 'scalar equation' versus 'vector equation' for planes in three-space,

formulas for distance from point to line (in two-space) or plane (in three-space), distance between two *skew* lines in three-space, etc.

6. Quadric surfaces: completing the square

Normalized form

$$\pm\left(\frac{x-x_o}{a}\right)^2 \pm\left(\frac{y-y_o}{b}\right)^2 \pm\left(\frac{z-z_o}{c}\right)^2 = 1$$

- ellipsoid if all plusses
- hyperboloid of *one* sheet if *one* minus
- hyperboloid of *two* sheets if *two* minuses
- empty set if three minuses

Degenerate cases: The first ones are **cones**

$$\pm\left(\frac{x-x_o}{a}\right)^2 \pm\left(\frac{y-y_o}{b}\right)^2 \pm\left(\frac{z-z_o}{c}\right)^2 = 0$$

Next are surfaces degenerate in a different sense: **paraboloids:**

$$\pm\left(\frac{x-x_o}{a}\right)^2 \pm\left(\frac{y-y_o}{b}\right)^2 + \frac{z-z_o}{c} = 0$$

- *same* sign: elliptic paraboloid
- *opposite* signs: hyperbolic paraboloid

Even more degenerate: **Quadric cylinders:** these occur when one of the variables x, y, z does not appear, but the equation is otherwise non-degenerate: here z does not appear

$$\pm\left(\frac{x-x_o}{a}\right)^2 \pm\left(\frac{y-y_o}{b}\right)^2 = 1$$

is **elliptic cylinder** and, even more degenerate,

$$\pm\left(\frac{x-x_o}{a}\right)^2 + \frac{y-y_o}{b} = 0$$

is a **parabolic cylinder**.

Derivatives of vector-valued functions of scalar variables: parametrized curves in two-space and three-space.

A **vector-valued function** $\vec{f}(t)$ is simply an ordered pair or ordered triple (or ...) of 'ordinary' functions. That is, a two-dimensional vector-valued function $\vec{f}(t)$ is just a function of the form

$$\vec{f}(t) = (f_1(2), f_2(t))$$

where f_1, f_2 are 'ordinary' numerical-valued functions of t. And a three-dimensional vector-valued function $\vec{f}(t)$ is just a function of the form

$$\vec{f}(t) = (f_1(2), f_2(t), f_2(t))$$

where f_1, f_2, f_3 are 'ordinary' numerical-valued functions of t. The functions $f_1, f_2, \ldots$ are the **components** of $\vec{f}$.

The **derivative** of a vector-valued function $\vec{f} = (f_1, f_2, \ldots)$ is just the vector of *plain* derivatives of its *components:* this is

$$\frac{d\vec{f}}{dt} = \vec{f}'(t) = (f_1', f_2', \ldots) = (\frac{df_1}{dt}, \frac{df_2}{dt}, \ldots)$$

So there's no new procedure here: differentiation of vector-valued functions is just several differentiations of 'ordinary' functions.

parametric equations for curves, parameter, example of *helix*

derivative of $\vec{f}(t)$ with respect to t as **tangent vector to curve**. And integral, too. **sum rule, 3 product rules, chain rule**.

Trick with length in terms of inner product: for example, if length of $\vec{r}$ is constant, then the tangent vector is orthogonal to it, for all t.

---

7. Arc length and curvature

First, the formula for **arc length** comes from the Pythagorean theorem, and is

$$\text{length of arc parametrized by } f\text{, from } a \text{ to } b = \int_a^b |\vec{f}'(t)|\, dt$$

Most of the integrals which arise in this manner are difficult or impossible to 'do' in symbolic terms. But *numerical* integration is always possible.

For *technical* or *theoretical* reasons, it is necessary to know that a given curve can be (re)parametrized in such a manner so that *the parameter is arc length*, at least in the sense that the *rates of change* of the two things are identical. (After all, actual measurement of arc length depends on where you start). This *reparametrization by arc length* is a necessary and useful thing. It is *not* useful for *computations*, however.

The first important observation is that, for a curve parametrized by a vector-valued function $\vec{f}$, a tangent vector to this curve, at the point $\vec{f}(t_o)$, is simply the derivative $\vec{f}'(t_o)$. Then, in the same way we make a *unit* vector from *any* non-zero vector, the **unit tangent vector** to the curve parametrized by $\vec{f}$ is

$$\vec{T} = \frac{\vec{f}'}{|\vec{f}'|}$$

The 'theoretical' definition of the **curvature** κ of $\vec{f}$ is

$$\kappa = \left|\frac{d\vec{T}}{ds}\right|$$

This is *not a good formula for computations*, since in practical terms it is unreasonable to expect to be able to get a *useful formula* for the reparametrization by arc length.

After going through some symbolic computations (see below), we obtain the *good general formula for curvature:*

$$\kappa = \frac{|\vec{f}' \times \vec{f}''|}{|\vec{f}'|^3} \quad \text{(general case)}$$

Still, this formula requires taking a cross product, which we know by now is a bit time-consuming. Happily, in simpler special circumstances there are simpler formulas. The *simplest case* is that where a *plane curve* is described by writing $y = f(x)$, that is, the vertical coordinate as a function of the first coordinate: in this case, the formula for curvature simplifies a lot, to be just

$$\kappa = \frac{|f''|}{[1 + (f')^2]^{3/2}} \quad \text{(for curve } y = f(x)\text{)}$$

In an intermediate case, that of a *parametrized plane curve* where both x, y are functions of a parameter t, we have

$$\kappa = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{[\dot{x}^2 + \dot{y}^2]^{3/2}} \quad \text{(for } x = x(t) \text{ and } y = y(t)\text{)}$$

where the dots denote differentiation with respect to t.

Then we use

$$\frac{d\vec{T}}{dt} = \frac{d\vec{T}}{ds}\frac{ds}{dt}$$

to obtain

$$\kappa = \left|\frac{d\vec{T}}{dt}\right| = \left|\frac{d\vec{T}/ds}{ds/dt}\right|$$

Then, using

$$|\vec{f}'| = \frac{ds}{dt}$$

we obtain

$$\kappa = \frac{|\vec{T}'|}{|\vec{r}'|}$$

But these formulas are not as practical as

$$\kappa = \frac{|\vec{f}' \times \vec{f}''|}{|\vec{f}'|^3}$$

The **principal unit normal** $\vec{N}$: since the unit tangent vector $\vec{T}$ always has length 1, its derivative $\vec{T}'$ is always *orthogonal* to it. Then

$$\frac{\vec{T}'}{|\vec{T}'|}$$

is the *principal unit normal.*

The **osculating plane** is the plane through the tangent and principal normal vectors.

The **unit binormal** is $\vec{B} = \vec{T} \times \vec{N}$. Check that $d\vec{B}/ds$ is orthogonal to $\vec{B}$, and perpendicular to $\vec{T}$, so is

$$\frac{d\vec{B}}{ds} = -\tau(s)\vec{N}$$

for **torsion** τ. We have

$$\tau = \frac{(\vec{f}' \times \vec{f}'') \cdot \vec{f}'''}{|\vec{f}' \times \vec{f}''|^2}$$

---

8. Velocity, acceleration, tangential and normal components

The point is that if $\vec{r}(t)$ is *position* at time t, then

$$\vec{v}(t) = \vec{r}'(t) = \dot{\vec{r}}(t)$$

is *velocity* at time t, and

$$\vec{a}(t) = \dot{\vec{v}}(t) = \ddot{\vec{r}}(t)$$

is *acceleration* at time t. And **speed** is the length of the velocity vector.

Example: Near the earth's surface, disregarding air resistance, the acceleration due to gravity is $-32\vec{j}$, if we use a coordinate system in which the y-axis points *upward*. This allows study of **projectiles near the earth's surface** *ignoring air resistance*. It is easy to derive the formula

$$\vec{s}(t) = -16t^2\vec{j} + t\vec{v_o} + \vec{s_o}$$

where t is *time*, $\vec{s}(t)$ is **position** at timte t, $\vec{v_o}$ is **initial velocity**, and $\vec{s_o}$ is **initial position**. This simple and easily-derived formula suffices to solve many problems in this simplest projectile model.

In more general situations, and for various reasons, one might want to know how much of the acceleration is in the direction of motion, and how much is 'to the side'. That is, we want to know the **tangential** and **normal** components of velocity a_T and a_N, respectively. That is, we want to write

$$\vec{a} = a_T\vec{T} + a_N\vec{N}$$

where $\vec{T}$ is the unit tangent and $\vec{N}$ is the principal unit normal vector. It turns out that

$$a_T = \frac{\vec{r}' \cdot \vec{r}''}{|\vec{r}'|}$$

$$a_N = \frac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'|}$$

9. Cylindrical and spherical coordinates

The real point here is just that there are formulas that allow a person to change back and forth among these three often-used systems:

Cylindrical coordinates (r, θ, z) are expressed in terms of *rectangular (usual)* coordinates by

$$r = \sqrt{x^2 + y^2} \qquad \theta = \arctan(y/x) \qquad (\text{ and } z = z)$$

Going in the other direction,

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad (\text{ and } z = z)$$

The cylindrical coordinates are just like *polar coordinates* in the plane, with the extra z along for the ride.

Spherical coordinates are often denoted by (ρ, θ, φ). There is the obvious ambiguity about which of the θ and φ gets used how... Letting θ be the **azimuthal angle**, the point (ρ, θ, φ) in spherical coordinates becomes gives the point (x, y, z) in rectangular coordinates where x, y, z are:

$$x = \rho\cos\theta\sin\varphi$$

$$y = \rho\sin\theta\sin\varphi$$

$$z = \rho\cos\varphi$$

Going in the opposite direction: expressing the spherical coordinates in terms of rectangular: for a point (x, y, z) in rectangular coordinates, the corresponding point (ρ, θ, φ) in spherical coordinates is given by

$$\rho = \sqrt{x^2 + y^2 + z^2} \qquad \varphi = \arccos\frac{z}{\sqrt{x^2 + y^2 + z^2}} \qquad \theta = \arctan y/x$$

A point (ρ, θ, φ) in *spherical coordinates* becomes (r, θ, z) in cylindrical coordinates with

$$z = \rho\cos\varphi$$

θ stays the same (This is a virtue of letting φ be the azimuthal angle!)

$$r = \rho\sin\varphi$$

Going in the other direction, the point (r, θ, z) in cylindrical coordinates becomes (ρ, θ, φ) in spherical coordinates, with

$$\rho = \sqrt{r^2 + z^2}$$

θ stays the same

$$\varphi = \arctan\frac{z}{r}$$

10. Level curves

and **level surfaces**

Level curves of a function $f(x, y)$ of *two* variables are *plane* curves of the form

$$f(x, y) = \text{ constant}$$

The qualitative nature of the curve may change considerably depending on which *particular* constant occurs on the right-hand side. In fact, we have to be a little careful because in some cases we'd just get a *point* or even *the empty set* rather than a *curve*. Still, the things are called *level curves*, even acknowledging that we have no actual guarantee that what we get is a *curve*.

Level surfaces of a function $f(x, y, z)$ of *three* variables are *space* curves of the form

$$f(x, y, z) = \text{ constant}$$

The precise nature of the curve may change somewhat depending on which *particular* constant occurs on the right-hand side. As in the case of level curves, we have to be a little careful because in some cases a level 'surface' is not a surface at all, but can be a *curve*, a *point*, or even *the empty set*, rather than a *surface*. Still, the things are called *level surfaces*, even acknowledging that we have no actual guarantee that what we get is a *surface*.

11. Limits and continuity in several variables:

Now, by contrast to what we'd just been doing, we'll look at **scalar-valued functions of vector variables**, rather than *vector functions of scalar variables* as above.

For our purposes, we mostly want to use our *intuition* about limits and continuity. In any case, for functions of more than one variable, things get out of hand almost right away: questions about limits are hard to answer, and there is really no 'systematic procedure' to answer even the ones that *are* answerable.

Evaluation by 'plugging-in': This amounts to invocation of continuity of the function at the limit point. Unless someone is trying to trick you, this should always be tried first: if nothing obviously bad happens then you probably got the right answer.

Evaluation by cancelling and then 'plugging-in': This issue already arises in *one variable* calculus, and is really what is hidden inside L'Hospital's rule and such things.

Evaluation by 'sandwiching' between simpler things: This is not a systematic method, but illustrates a little bit about an important subtle point.

12. Partial Derivatives:

Let $f(x, y, \ldots)$ be a function of 'several' variables. The **partial derivative** of f *with respect to* x can be written in several different ways:

$$\frac{\partial f}{\partial x} = f_x = D_1 f = \partial_1 f$$

and is *defined* to be

$$f_1(x, y, z) = \lim_{h \to 0} \frac{f(x+h, y, \ldots) - f(x, y, \ldots)}{h} = \lim_{x_1 \to x} \frac{f(x_1, y, \ldots) - f(x, y, \ldots)}{x_1 - x}$$

That is, the *other* variables $y, \ldots$ *do not change* while the limit involving x is being evaluated. Similarly, the **partial derivative** of f *with respect to* y can be denoted in several ways:

$$\frac{\partial f}{\partial y} = f_x = D_2 f = \partial_2 f$$

and is *defined* to be

$$f_2(x, y, z) = \lim_{h \to 0} \frac{f(x, y+h, \ldots) - f(x, y, \ldots)}{h} = \lim_{y_1 \to y} \frac{f(x, y_1, \ldots) - f(x, y, \ldots)}{y_1 - y}$$

That is, the *other* variables $x, \ldots$ *do not change* while the limit involving y is being evaluated.

There are various traditional and possibly misleading descriptions about *'holding the other variables fixed'* or *'viewing the other variables as constants'* or some such things, but there is really nothing mystical about it: whatever values the other *inputs* to the function have, we simply don't *change* them.

Very important: Just as in single-variable calculus, the *definition* of partial derivative given above is essentially *never* used to actually compute anything. Rather, just as in the one-variable case, we have a collection of standard formulas which can be used to compute derivatives *almost effortlessly*, at least by comparison to trying to use the definitions directly.

And the formulas that are used in 'single-variable' calculus are immediately applicable in 'several-variable' calculus, because the derivative is defined as a certain kind of limit which is *just the same. So there aren't any new formulas.*

Caution: The variable x is *not always* the first input to a function, nor is y always the *second* input, etc.! Although it is *often* the case that x is the first input, y is the second, and the third if any is z, *this cannot be depended upon.* Therefore, for example, the notation $\partial f/\partial x$ is not perfectly reliable! Rather, it is better to say *which input* the derivative is taken with respect to: we might say *derivative with respect to the first input*, or *derivative with respect to the second input*, and so on.

This reference to *which input* rather than to a *name of a variable* is more reliable, and is the sense of the notation f_1, f_2, f_3, etc., meaning (respectively) derivative with respect to the first, second, third, ... inputs of the function f.

But then if you *are* going to use f_2 to mean the partial derivative of f with respect to its second input, it would be bad to simultaneously try to use this notation for the the second *component* of a vector, eh?

Further caution: Also, although very often the first input is x, and so on, there is no rule about this, and it may in fact be desirable to use *other inputs.* Thus, if there is any possibility for confusion, a person should **write out the inputs for a function**. For example, we might *want* to consider

$$f(y, z, x)$$

rather than $f(x,y,z)$. That is, the 'popular' variables are not used in the usual order. Whenever such is done it is obviously necessary to spell it out.

And **higher partial derivatives** are just repeated partial derivatives: for example

$$f_{11} = f_{xx} = \frac{\partial^2 f}{(\partial x)^2} = \frac{\partial}{\partial x}\frac{\partial}{\partial x} f$$

$$f_{12} = f_{xy} = \frac{\partial^2 f}{\partial ; x\, \partial y} = \frac{\partial}{\partial x}\frac{\partial}{\partial y} f$$

$$f_{21} = f_{yx} = \frac{\partial^2 f}{\partial ; y\, \partial x} = \frac{\partial}{\partial x}\frac{\partial}{\partial y} f$$

$$f_{22} = f_{yy} = \frac{\partial^2 f}{\partial ; y\, \partial y} = \frac{\partial}{\partial y}\frac{\partial}{\partial y} f$$

$$f_{13} = f_{xz} = \frac{\partial^2 f}{\partial ; x\, \partial z} = \frac{\partial}{\partial x}\frac{\partial}{\partial z} f$$

$$f_{23} = f_{yz} = \frac{\partial^2 f}{\partial ; y\, \partial z} = \frac{\partial}{\partial y}\frac{\partial}{\partial z} f$$

and so on. The same **cautions** apply, in terms of being too confident that x is always the first input, and so on.

But, on the other hand, to avoid crazily burdensome notation, we must be willing to risk notation confusion!

---

13. Gradient, directional derivatives

The **gradient** of a *scalar-valued* function $f(x,y,z)$ is simply the *vector* of all the (first) partial derivatives:

$$\nabla f(x,y,z) = (\frac{\partial f}{\partial x}(x,y,z),\ \frac{\partial f}{\partial y}(x,y,z),\ \frac{\partial f}{\partial z}(x,y,z)) = (f_x, f_y, f_y)$$

The **directional derivative** of a function $f(x,y,z)$ in a direction $\vec{v}$ is just the dot product of ∇f with the *unit vector* $\frac{\vec{v}}{|\vec{v}|}$ in the direction of $\vec{v}$:

$$\text{directional derivative in direction of } \vec{v} = \frac{\vec{v}}{|\vec{v}|} \cdot \nabla f(x,y,z)$$

This isn't the actual definition, but rather is the formula that you use to *compute* the directional derivative.

The **direction of greatest increase** of a function $f(x,y,z)$ at a point (x_o, y_o, z_o) is in the direction of the gradient (*evaluated at that point*):

$$\text{direction of greatest increase of } f \text{ at } (x_o, y_o, z_o) = \nabla f(x_o, y_o, z_o)$$

If someone insists that a *direction* be given only by a *unit* vector, then we would say

$$\text{direction of greatest increase of } f \text{ at } (x_o, y_o, z_o) = \frac{\nabla f(x_o, y_o, z_o)}{|\nabla f(x_o, y_o, z_o)|}$$

Similarly, the direction of greatest *decrease* is in the *opposite* direction, that is, in the direction $-\nabla f(x_o, y_o, z_o)$.

Then, the **amount of greatest increase** is obtained by taking the *directional derivative* in the *direction* of greatest increase, which gives

$$\text{amount of greatest increase at } (x_o, y_o, z_o) \ =$$

$$= \frac{\nabla f(x_o, y_o, z_o)}{|\nabla f(x_o, y_o, z_o)|} \cdot \nabla f(x_o, y_o, z_o) = |\nabla f(x_o, y_o, z_o)|$$

That is, the *amount* of greatest increase is simply the length of the gradient, while the *direction* of greatest increase is in the *direction* of the gradient.

14. Perpendicular vectors to level surfaces

The only point here is that *the gradient* $\nabla f(x_o, y_o, z_o)$ *is perpendicular to a level surface described by* $f(x, y, z) = c$ *at the point* (x_o, y_o, z_o) (which is presumed to be *on* that level surface). That is, we have a *very easy* way to find vectors that are perpendicular ('normal') to surfaces described as *level* surfaces.

Sometimes is it necessary to *rewrite* equations describing curves or surfaces to make them look like *level* curves and surfaces. For example, if a surface is described as being the set of (x, y, z) so that

$$z = f(x, y)$$

for some given function f of two variables, then we need to rewrite this as

$$f(x, y) - z = 0$$

to make it look like a level surface of the new function F defined by

$$F(x, y, z) = f(x, y) - z$$

If we don't do this, various fatal boo-boos will creep in.

15. Linear Approximation

... or, in archaic terminology, **'approximation by differentials'**

The point is that, for a function $f(x, y, z)$ of several variables (for example, *three*, as in this case), for 'small changes' dx, dy, dz in the three inputs,

$$f(x + dx, y + dy, z + dz)$$

$$\approx f(x, y, z) + \frac{\partial f}{\partial x}(x, y, z)\, dx + \frac{\partial f}{\partial y}(x, y, z)\, dy + \frac{\partial f}{\partial z}(x, y, z)\, dz$$

where we use the symbol $\approx$ mean *approximately equal to.* We will let the latter phrase have its obvious informal meaning, without struggling to formalize it.

Written in terms of the *gradient*

$$\nabla f(x, y, z) = (\frac{\partial f}{\partial x}(x, y, z),\ \frac{\partial f}{\partial y}(x, y, z),\ \frac{\partial f}{\partial z}(x, y, z))$$

the formula is

$$f(x + dx, y + dy, z + dz) \approx f(x, y, z) + \nabla f(x, y, z) \cdot (dx, dy, dz)$$

Note that sometimes people write

$$(\Delta x, \Delta y, \Delta z)$$

instead of (dx, dy, dz), and then the formulas look like

$$f(x + \Delta x, y + \Delta y, z + \Delta z)$$

$$\approx f(x, y, z) + \frac{\partial f}{\partial x}(x, y, z)\, \Delta x + \frac{\partial f}{\partial y}(x, y, z)\, \Delta y + \frac{\partial f}{\partial z}(x, y, z)\, \Delta z$$

$$\approx f(x + \Delta x, y + \Delta y, z + \Delta z) \approx f(x, y, z) + \nabla f(x, y, z) \cdot (\Delta x, \Delta y, \Delta z)$$

The notion of 'approximation' implied here is *not* one that is very useful in getting high-precision numerical approximations. Rather, the virtue is the *simplicity* of the expressions, and the nice systematic way that the approximating expressions depends upon the inputs. That is, *these formulas give not just approximations one number at a time, but rather give a whole family of approximations fitting into a nice pattern.*

Prior to the widespread availability of computing machines, it seemed reasonable to ask questions like *Approximate* $\sqrt{17}$ *by 'differentials'.* This type of question didn't really make sense 20 years ago, and it makes almost no sense *now*: why not just hit the buttons '1', '7', and 'sqrt' on your calculator and get the answer to 14 decimals? Indeed!

The idea was that if we are computing *by hand*, there is no instantaneous way to get $\sqrt{17}$, by contrast to the fact that we 'recognize' that $\sqrt{16} = 4$ and $\sqrt{25} = 5$. In fact, with $f(x) = \sqrt{x}$, not only can we 'easily' compute $f(16) = \sqrt{16}$, but also $f'(16) = \frac{1}{2}\frac{1}{\sqrt{16}}$. Thus, in this example, we can 'easily' compute *by hand* all the materials that we need to plug into the formula above:

$$\sqrt{17} = \sqrt{16 + 1} = f(16 + 1) \approx f(16) + f'(16) \cdot 1 = \sqrt{16} + \frac{1}{2}\frac{1}{\sqrt{16}} = 4 + \frac{1}{8} = 4.125$$

By contrast, from a calculator we find that

$$\sqrt{17} = 4.1231056256176...$$

So we're off by .002, which is not terrible, but still pathetic in a way.

On the other hand, suppose we wanted not *numbers* but *patterns*, in an example like the following question: *approximate* $\sqrt{x+1} - \sqrt{x}$. This is not a numerical question, and while you certainly can do a lot of *numerical examples* with your calculator, all those numbers would only hint at what the general pattern might be, if any.

It is in such a scenario that the linear approximation idea is really useful, and says something with worthwhile content. Letting $f(x) = \sqrt{x}$, we have

$$\sqrt{x+1} - \sqrt{x} = f(x+1) - f(x) \approx [f(x) + f'(x) \cdot 1] - f(x) = f'(x) = \frac{1}{2}\frac{1}{\sqrt{x}}$$

It is true that a similar result can be achieved by some algebra tricks (also worth knowing):

$$\sqrt{x+1} - \sqrt{x} = \frac{\sqrt{x+1} + \sqrt{x}}{\sqrt{x+1} + \sqrt{x}}(\sqrt{x+1} - \sqrt{x})$$

$$= \frac{(x+1) - x}{\sqrt{x+1} + \sqrt{x}} = \frac{1}{\sqrt{x+1} + \sqrt{x}}$$

This already gives an idea of what happens when x is 'large'. Its disadvantage is that the expression given, while perfectly accurate, is significantly more complicated than the $\frac{1}{2}\frac{1}{\sqrt{x}}$ obtained by linear approximation.

In general, it may be worthwhile to use a linear approximation when *simplicity* is very important. Especially when the expected answer itself is not intended to be terribly precise, it is pointless to give extra decimal places. Or, in other circumstances, it may be that the data you are given is itself not very accurate, so that making very delicate deductions from it is silly.

Five numbers between 10 and 15 are rounded off to the nearest 1/10 and then multiplied together. *Estimate the error:*

Note that when something is rounded off to the nearest $\frac{1}{10}$, the error introduced is no worse than $\frac{1}{2}\frac{1}{10}$, since we never have to move by a whole 1/10: for example, from 3.16 we go up to 3.2, from 3.15 up to 3.2, but from 3.14 go *down* to 3.1.)

Also, notice that in general to measure error we must use *absolute values*, rather than just subtract. By this is meant that if A is an approximation to V, then the error is

$$\text{error } = |A - V|$$

This way, to say 'error $< \frac{1}{100}$' really means something. By contrast, if we just took the error to be $A - V$ and said 'error $< \frac{1}{100}$' it could conceivably be that the error was -10000 since, after all, $-10000 < \frac{1}{100}$. We don't want this.

Let the 5 numbers be x_1, x_2, x_3, x_4 and x_5. Let their rounded-off versions be $x_i + h_i$ (or use δ_i instead of h_i if you like). Let

$$f(x_1, x_2, x_3, x_4, x_5) = x_1x_2x_3x_4x_5$$

We see that

$$\nabla f(x_1, x_2, x_3, x_4, x_5) =$$

$$(x_2x_3x_4x_5,\ x_1x_3x_4x_5,\ x_1x_2x_4x_5,\ x_1x_2x_3x_5,\ x_1x_2x_3x_4)$$

Then the difference between the actual product and the product of the rounded-off numbers, in absolute value, is

$$
\begin{aligned}
&|(x_1+h_1)\dots(x_5+h_5) - x_1\dots x_5| \\
&= |f(x_1+h_1,\dots,\dots,x_5+h_5) - f(x_1,\dots,x_5)| \\
&\approx |\nabla f(x_1,x_2,\dots,x_5)\cdot(h_1,h_2,\dots,h_5)| \\
&= |(x_2x_3x_4x_5,\ \dots,x_1x_2x_3x_4)\cdot(h_1,\ h_2,\ \dots,h_5)| \\
&= |h_1x_2x_3x_4x_5 + x_1h_2x_3x_4x_5 + x_1x_2h_3x_4x_5 + x_1x_2x_3h_4x_5 + x_1x_2x_3x_4h_5|
\end{aligned}
$$

Using the triangle inequality, as would be the usual kind of thing to do here,

we have

$$
\begin{aligned}
&|h_1x_2x_3x_4x_5 + x_1h_2x_3x_4x_5 + x_1x_2h_3x_4x_5 + x_1x_2x_3h_4x_5 + x_1x_2x_3x_4h_5| \\
&\leq |h_1x_2x_3x_4x_5| + |x_1h_2x_3x_4x_5| + |x_1x_2h_3x_4x_5| + |x_1x_2x_3h_4x_5| + |x_1x_2x_3x_4h_5|
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
\text{error } &\leq |h_1x_2x_3x_4x_5| + |x_1h_2x_3x_4x_5| \\
&+|x_1x_2h_3x_4x_5| + |x_1x_2x_3h_4x_5| + |x_1x_2x_3x_4h_5|
\end{aligned}
$$

The hypothesis that we round off to the nearest 1/10 assures that $|h_i| \leq \frac{1}{20}$. And each of the $x_1,\dots,x_5$ is between 10 and 15. Putting these two things together allows us to *estimate* each term, meaning that we can say what the **worst-case scenario** would be: for example,

$$|h_1x_2x_3x_4x_5| \leq (\frac{1}{20})(15)(15)(15)(15)$$

since taking $h_1 = \frac{1}{20}$ and $x_2 = x_3 = x_4 = x_5 = 15$ makes this term the largest it can be.

Doing this *estimation* on all five terms at once gives

$$\text{error } \leq 5\cdot(\frac{1}{20}\cdot 15^4)$$

---

16. Chain Rule

Just as with the Chain Rule in single-variable calculus, we need to have a systematic way to take derivatives of functions defined as *composite* functions, built up out of simpler functions into possibly very messy things. In fact, if one thinks about it the right way, the several-variables chain rule *is* obtained by repeated application of the single-variable one. Just as 'partial derivatives' are really just derivatives in a different context, the several-variable chain rule is not really new, either.

Just as in the one-variable story, sometimes if all the functions involved are polynomials or some other explicit and 'elementary' things, then you can avoid official invocation of any chain rule at all by just writing out the whole thing in morbid detail. But sometimes this approach, even if possible, is just too labor-intense, as we'll see in examples below.

There is another hazard here, too, which involves failure to look behind the symbols at the *meaning* of the formulas. This problem can be bad-enough already in one-variable calculus, but the potential for trouble is much greater with several variables. So, for starters, it is *not* true that x is always the first input of a function, nor that y is always the second, etc.

First let's review the **single-variable chain rule**. In symbols: if $F(x)$ is a function of input x defined as

$$F(x) = f(g(x))$$

then

$$F'(x) = f'(g(x)) \cdot g'(x)$$

And recall that sometimes the composite function $f(g(x))$ is written as

$$f(g(x)) = (f \circ g)(x)$$

which gives us a way of talking about the function $f \circ g$ without telling what input we plan to use in it.

Notice in this scenario that the input to f' is not x, but $g(x)$. It is important to stop and realize that the symbol f' does not mean 'the derivative of f with respect to x', but rather 'the derivatie of f with respect to its input'. This is always true, but we tolerate writing df/dx and such things because *very often* x really is the input to f. Just to drive this point into the ground, let's realize that

$$f'(g(z)) = \lim_{\delta \to 0} \frac{f(g(x) + \delta) - f(g(x))}{\delta}$$

and this δ is *not* any more affiliated with x than it is with $g(x)$ or any other thing. Instead, δ is something added to *the input*, whatever it may turn out to be.

Another little sticking-point is that some of the most familiar functions in our repertory don't have very good names. For example, the function $f(x) = x^2$ is often just called 'the x^2 function'. Usually there's no confusion, but at the bottom line we can complain that *just telling the output of a function doesn't describ the function very well, unless we know the input also!*

Example: Let $f(x) = (1 + x^2)^{100}$. It is not reasonable to multiply this out by hand. Here we immediately encounter the phenomenon just mentioned: if you'd only just started doing this stuff you might not recognize that the right-hand side is a *composite function*, because the functions there don't have 'names'. One the other hand, if we just use words rather than symbols, we can write

$$f(x) = \text{take-hundredth-power} \circ \text{square-and-add-1}(x)$$

In symbols, we can *give* these nameless functions single-letter names: let $g(x) = x^2 + 1$ and $h(x) = x^{100}$. Then

$$f(x) = h(g(x))$$

Also there is an opportunity to misunderstand: the function description $h(x) = x^{100}$ has nothing really to do with the 'x', but rather says that the function h has the effect of raising its input to the hundredth power and giving that hundredth power back as output. So $h(g(x)) = g(x)^{100}$. The nasty mistake would be to think that *no matter what the input* still $h(*) = x^{100}$.

Then the chain rule gives

$$\frac{d}{dx}(1+x^2)^{100} = 100(1+x^2) \cdot \frac{d}{dx}(1+x^2) = 100(1+x^2) \cdot 2x$$

If the pitfalls are avoided then it's easy.

The basic chain rule formula in one variable is, in reality, just the 'atom' from which much more complicated formulas can be obtained by repeated application of the principle. For example, in symbols, let

$$F(x) = f(g(h(x)))$$

Then, in steps,

$$\frac{d}{dx}F(x) = f'(g(h(x))) \cdot \frac{d}{dx}g(h(x)) = f'(g(h(x))) \cdot g'(h(x)) \cdot h'(x)$$

by applying the chain rule twice.

Example: Find $\frac{d}{dx}(1+\cos(e^x))^{100}$: by the chain rule, applied repeatedly,

$$\frac{d}{dx}(1+\cos(e^x))^{100} = 100(1+\cos(e^x))^{99} \cdot \frac{d}{dx}(1+\cos(e^x))$$

$$= 100(1+\cos(e^x))^{99} \cdot (-\sin(e^x)) \cdot \frac{d}{dx}e^x = 100(1+\cos(e^x))^{99} \cdot (-\sin(e^x)) \cdot e^x$$

A symbolic presentation of the several-variable chain rule

Let

$$F(x,y,z,\ldots) = f(u,v,w,\ldots)$$

where the dots inside F indicate that there may be more inputs to F than those visibly listed, and where u, v, w and so on are functions which depend in some unspecified way on $x, y, z, \ldots$. Then the **chain rule** asserts that

$$\frac{\partial}{\partial x}F(x,y,z,\ldots)$$

$$= f_1(u,v,w,\ldots) \cdot \frac{\partial u}{\partial x} + f_2(u,v,w,\ldots) \cdot \frac{\partial v}{\partial x} + f_3(u,v,w,\ldots) \cdot \frac{\partial w}{\partial x} + \ldots$$

where the subscript i refers to the partial derivative of f with respect to the i^{th} input. Similarly,

$$\frac{\partial}{\partial y}F(x,y,z,\ldots)$$

$$= f_1(u,v,w,\ldots) \cdot \frac{\partial u}{\partial y} + f_2(u,v,w,\ldots) \cdot \frac{\partial v}{\partial y} + f_3(u,v,w,\ldots) \cdot \frac{\partial w}{\partial y} + \ldots$$

$$\frac{\partial}{\partial z}F(x,y,z,\ldots)$$

$$= f_1(u, v, w, \ldots) \cdot \frac{\partial u}{\partial z} + f_2(u, v, w, \ldots) \cdot \frac{\partial v}{\partial z} + f_3(u, v, w, \ldots) \cdot \frac{\partial w}{\partial z} + \ldots$$

The idea here is that $x, y, z, \ldots$ are *independent* of each other. In particular,

$$\frac{\partial y}{\partial x} = 0 \qquad \frac{\partial x}{\partial y} = 0$$

$$\frac{\partial z}{\partial x} = 0 \qquad \frac{\partial x}{\partial z} = 0$$

$$\frac{\partial y}{\partial z} = 0 \qquad \frac{\partial z}{\partial y} = 0$$

and so on.

In principle, if we read these all formulas astutely, they potentially contain all that there is to say about the chain rule. But it is wise to work up to this gradually.

Several-variable chain rule reduced to single-variable

The one-variable version is enough for many simple several-variable problems:

Example: Define $f(x, y) = F(x + y, x - y)$ where $F(a, b) = a^2 b^3$. Find $\partial f / \partial x$.

Let's do this one by *writing it out*. This amounts simply to rewriting F with inputs $x + y$ and $x - y$ instead of a and b:

$$f(x, y) = F(x + y, x - y) = (x + y)^2 (x - y)^3$$

If one wanted to, one could multiply this out: it is

$$f(x, y) = x^5 - 2x^3 y^2 - x^4 - 2x^2 y^3 + x^4 y - y^5$$

Then

$$\frac{\partial f}{\partial x} = 5x^4 - 6x^2 y^2 - 4x^3 - 4xy^3 + 4x^3 y - 0$$

Done.

Example: As a variation on the previous: Define $f(x, y) = F(x + y, x - y)$ where $F(a, b) = a^5 b^6$. Find $\partial f / \partial x$.

Now it's less appealing to use the simple-minded approach of the previous example, since you probably don't want to take the fifth or sixth powers of $x + y$ or $x - y$. Still, we start by rewriting F with inputs $x + y$ and $x - y$ instead of a and b:

$$f(x, y) = F(x + y, x - y) = (x + y)^5 (x - y)^6$$

If one really wanted to, one could multiply this out, but it would be a large expression. Instead of that, let's just use the product rule and the chain rule that we already know from one-variable calculus:

$$\frac{\partial}{\partial x} f(x, y) = [\frac{\partial}{\partial x}(x + y)^5](x - y)^6 + (x + y)^5 [\frac{\partial}{\partial x}(x - y)^6]$$

$$= 5(x + y)^4 \frac{\partial (x + y)}{\partial x}(x - y)^6 + (x + y)^5 \cdot 6(x - y)^5 \frac{\partial (x - y)}{\partial x}$$

$$= 5(x + y)^4 (x - y)^6 + (x + y)^5 \cdot 6(x - y)^5$$

This can be further simplified, as well, since the two summands have so many common factors: it becomes

$$(x+y)^4(x-y)^5)[5(x-y)+6(x+y)] = (x+y)^4(x-y)^5)(11x+y)$$

Done.

Example: Define $f(x,y) = G(u(x,y), v(x,y), w(x,y))$ where $u(a,b) = ab$, $v(a,b) = a^2+b^2$, $w(a,b) = b$, and $G(a,b,c) = ab - c$. Find $\partial f/\partial x$.

Not only did the expressions get bigger and messier, but the names of the inputs used to describe the functions u, v, w, G are not the same as what we plan to use. For example, we are told that $u(a,b) = ab$, but we care about $u(x,y)$ instead. But the phrase $u(a,b) = ab$ should be read in a way which doesn't so much refer to a, b, but rather tells *what the function does to its inputs.* That is, since a is the first input and b is the second, we see that u just multiplies its two inputs together. So certainly $u(x,y) = xy$. Likewise, v evidently adds the squares of its two inputs, so $v(x,y) = x^2+y^2$. And the output of w is simply its second input, so $w(x,y) = y$.

And, similarly, we are told that $G(a,b,c) = ab - c$, but the actual inputs we plan to use are $u(x,y)$, $v(x,y)$, and $w(x,y)$. Reading the description of G *not* as being about a, b, c, but rather about what G does to its inputs, we see that it multiplies the first two together and subtracts the third from that product. Therefore,

$$G(u(x,y), v(x,y), w(x,y)) = u(x,y)v(x,y) - w(x,y)$$

We can do this by 'writing it out' again: First, we realize that the desired inputs to G are not a, b, c but, rather, $u(x,y), v(x,y), w(x,y)$. So we rewrite $G(a,b,c) = ab - c$ as

$$G(u(x,y), v(x,y), w(x,y)) = u(x,y)v(x,y) - w(x,y)$$

And instead of inputs a, b, each of u, v, w will have inputs x, y. So we rewrite $u(x,y) = xy$, $v(x,y) = x^2+y^2$, $w(x,y) = y$. Then

$$f(x,y) = G(u(x,y), v(x,y), w(x,y)) = u(x,y)v(x,y) - w(x,y) = (xy)(x^2+y^2) - y$$

This is easy to multiply out completely:

$$f(x,y) = x^3y + xy^3 - y$$

Then

$$\frac{\partial f}{\partial x}(x,y) = 3x^2y + y^3 - 0$$

Done.

Example: Let $f(x,y,z) = G(xy, yz, zx)$ where $G(a,b,c) = a + abc - bc$. Find $\partial f/\partial x$:

In the expression for G, replace a by xy, b by yz, and c by zx:

$$f(x,y,z) = G(xy, yz, zx) = (xy) + (xy)(yz)(zx) - (yz)(zx) = xy + x^2y^2z^2 - xyz^2$$

Then the partial derivative is easy to take:

$$\frac{\partial f}{\partial x}(x,y,z) = y + 2xy^2z^2 - yz^2$$

Done.

It is important to realize, though, that some problems *cannot* be so conveniently turned into stuff that looks like single-variable calculus. And sometimes even when it's possible it's not optimal.

A slightly more general pattern

The *atomic piece* of *one* 'fuller' several-variable chain rule is the following. Let

$$F(x,y,z) = f(u(x,y,z), v(x,y,z), w(x,y,z))$$

Then

$$\begin{aligned}\frac{\partial}{\partial x}F(x,y,z) &= f_1 \cdot \frac{\partial}{\partial x}u + f_2 \cdot \frac{\partial}{\partial x}v + f_3 \cdot \frac{\partial}{\partial x}w \\ &= f_1 \cdot u_1 + f_2 \cdot v_1 + f_3 \cdot w_1\end{aligned}$$

where we use the abbreviations

$$\begin{aligned}f_1 &= f_1(u(x,y,z), v(x,y,z), w(x,y,z)) \\ f_2 &= f_2(u(x,y,z), v(x,y,z), w(x,y,z)) \\ f_3 &= f_3(u(x,y,z), v(x,y,z), w(x,y,z)) \\ u_1 &= u_1(x,y,z) \\ v_1 &= v_1(x,y,z) \\ v_1 &= v_1(x,y,z)\end{aligned}$$

For a function ϕ a subscript ϕ_i means to take the partial derivative with respect to the i^{th} input. Even though it's maybe a bit annoying to set up these abbreviations explicitly, it is wise to do so.

Here it is important to realize that it is because of the simplicity of the situation that we have

$$\frac{\partial}{\partial x}u = u_1 \qquad \frac{\partial}{\partial x}v = v_1 \qquad \frac{\partial}{\partial x}w = w_1$$

That is, x really *is* the first input of each of u, v, w, which might not at all be the case in more complicated scenarios.

In the same set-up, with the same notational conventions,

$$\frac{\partial}{\partial y}F(x,y,z) = f_1 \cdot u_2 + f_2 \cdot v_2 + f_3 \cdot w_2$$

$$\frac{\partial}{\partial z}F(x,y,z) = f_1 \cdot u_3 + f_2 \cdot v_3 + f_3 \cdot w_3$$

But this sample-formula is misleading if it makes you think that there have to be three functions u, v, w, each taking three inputs, and so on. No such constraints are necessary at all. Using a slightly different notation to suggest an arbitrary number of inputs... Let

$$f(x,y,z,\ldots) = F(u(x,y,z,\ldots), v(x,y,z,\ldots), \ldots)$$

Then

$$\frac{\partial}{\partial x}F(x,y,z,\ldots) = f_1 \cdot u_1 + f_2 \cdot v_1 + f_3 \cdot w_1 + \ldots$$

$$\frac{\partial}{\partial y}F(x,y,z,\ldots) = f_1 \cdot u_2 + f_2 \cdot v_2 + f_3 \cdot w_2 + \ldots$$

$$\frac{\partial}{\partial z}F(x,y,z,\ldots) = f_1 \cdot u_3 + f_2 \cdot v_3 + f_3 \cdot w_3 + \ldots$$

and where we use abbreviations

$$f_1 = f_1(u(x,y,z,\ldots), v(x,y,z,\ldots),\ldots)$$

$$f_2 = f_2(u(x,y,z,\ldots), v(x,y,z,\ldots),\ldots)$$

$$f_3 = f_3(u(x,y,z,\ldots), v(x,y,z,\ldots),\ldots)$$

$$u_1 = u_1(x,y,z,\ldots)$$

$$v_1 = v_1(x,y,z,\ldots)$$

$$v_1 = v_1(x,y,z,\ldots)$$

and so on. *Done.*

And of course this 'simple' principle can be 'compounded' in the same way that the single-variable chain rule often has to be repeatedly applied. The open-ended-ness of the several-variable version makes it nearly impossible to have a single 'universal' formula to express this.

Example: Let

$$F(x,y) = f(u(x,y), v(a(x,y), b(x,y)))$$

Then, in steps, using similar abbreviations to above,

$$\frac{\partial}{\partial x}F(x,y) = f_1 \cdot \frac{\partial}{\partial x}u(x,y) + f_2 \cdot \frac{\partial}{\partial x}v(a(x,y), b(x,y))$$

$$= f_1 \cdot u_1 + f_2 \cdot [v_1 \cdot a_1 + v_2 \cdot b_1]$$

Done.

It is completely ok to have different depths of parentheses, for example.

Example: Let

$$F(x,y) = f(u(x,y), v(a(y,y), b(x,x,y)))$$

and find $\partial F/\partial x$. Note that here the inputs to the 'inner' functions are a bit funny: the inputs in a are *both* y, and the inputs to b are x, x, and y. This is legal. The fact that some people might be disturbed by it does not affect things, except that we have to be a little more careful in use of abbreviations. In steps, using similar abbreviations to above,

$$\frac{\partial}{\partial x}F(x,y) = f_1 \cdot \frac{\partial}{\partial x}u(x,y) + f_2 \cdot \frac{\partial}{\partial x}v(a(y,y), b(x,x,y))$$

$$= f_1 \cdot u_1 + f_2 \cdot [v_1 \cdot \frac{\partial}{\partial x}a(y,y) + v_2 \cdot \frac{\partial}{\partial x}b(x,x,y)]$$

$$= f_1 \cdot u_1 + f_2 \cdot v_1 \cdot 0 + f_2 \cdot v_2 \cdot [b_1 + b_2 + 0]$$

since $a(y,y)$ does not depend at all on x, and since $\partial y/\partial x = 0$ as well. *Done.*

Aggravating things even further:

Example: Let

$$F(x,y) = f(u(x,y), v(a(y,xy), b(x^2,x^3,y)))$$

and find $\partial F/\partial x$.

$$\frac{\partial}{\partial x}F(x,y) = f_1 \cdot \frac{\partial}{\partial x}u(x,y) + f_2 \cdot \frac{\partial}{\partial x}v(a(y,xy), b(x^2,x^3,y))$$

$$= f_1 \cdot u_1 + f_2 \cdot [v_1 \cdot \frac{\partial}{\partial x} a(y, xy) + v_2 \cdot \frac{\partial}{\partial x} b(x^2, x^3, y)]$$

$$= f_1 \cdot u_1 + f_2 \cdot v_1 \cdot [0 + a_2 \cdot \frac{\partial}{\partial x}(xy) + f_2 \cdot v_2 \cdot [b_1 \cdot \frac{\partial}{\partial x} x^2 + b_2 \cdot \frac{\partial}{\partial x} x^3 + 0]$$

$$= f_1 \cdot u_1 + f_2 \cdot v_1 \cdot [0 + a_2 \cdot y] + f_2 \cdot v_2 \cdot [b_1 \cdot 2x + b_2 \cdot 3x^2 + 0]$$

Done.

An explanatory story

The several-variable chain rule may look very different from the single-variable version, because there are so many terms spit out by it. In the one variable case even when the chain rule has to be repeatedly applied it only adds one thing at a time. This indicates some need for an explanation which would help in remembering the several-variable case.

One way of putting it into words is that we want to see how a composite function such as

$$F(x, y, \ldots) = f(u(a(x, y), b(y, y)), v(c(x, y, y), \ldots)$$

changes when x (or y) is changed. Of course we do not pretend to 'visualize' the whole thing at once, but only want to do things one intelligible step at a time. We can account for all the terms which occur in the following manner: find every occurence of x inside the parentheses and 'work outward' through the parentheses. Each occurence of x will give one 'term' in the expression for $\partial F/\partial x$.

This viewpoint may often be an effective bookkeeping device to make sure that no terms are overlooked.

Example: Let

$$F(x, y, \ldots) = f(u(a(x, y), b(x, x^2, y)), v(y), w(c(y, xy), d(x)))$$

and express $\partial F/\partial x$ in terms of f, u, v, w, a, b, c, d and their partial derivatives. To make sure that nothing is overlooked, it may be best to generate a term corresponding to each 'x' occuring, rather than trying to apply the chain rule step by step.

There are 5 places where and 'x' occurs: the first one is as the first input to a, which itself occurs as the first input to u, which itself occurs as the first input to f. This will yield the term

$$f_1 \cdot u_1 \cdot a_1$$

The next occurence of x is as the first input of b, which itself is the second input of u, which is the first input of f. This gives the term

$$f_1 \cdot u_2 \cdot b_1$$

The next occurence of x is *squared* as the second input to b, which itself is the second input of u, which is the first input of f. This gives the term

$$f_1 \cdot u_2 \cdot b_2 \cdot \frac{\partial x^2}{\partial x} = f_1 \cdot u_2 \cdot b_2 \cdot 2x$$

The next occurence of x is in the 'xy' which is the second input of c, which is the first input of w, which is the fourth input of f. This gives the term

$$f_4 \cdot w_1 \cdot c_2 \cdot \frac{\partial xy}{\partial x} = f_4 \cdot w_1 \cdot c_2 \cdot y$$

The next occurence of x is as the (only) input to d, which is the second input to w, which is the fourth input to f. This gives the term

$$f_4 \cdot w_2 \cdot d_1$$

Adding all these terms up, understanding that *the inputs are to be the same as in the statement of the problem*, we have

$$\frac{\partial F}{\partial x}$$
$$= f_1 \cdot u_1 \cdot a_1 + f_1 \cdot u_2 \cdot b_1 + f_1 \cdot u_2 \cdot b_2 \cdot 2x + f_4 \cdot w_1 \cdot c_2 \cdot y + f_4 \cdot w_2 \cdot d_1$$

Using symbolic methods when things get messy

Finally, we can see how to use these *symbolic* methods even when the functions have formulaic descriptions. For example, if the written-out formulas don't fit on a single line, or on a single page, you have almost no chance of successfully recopying them the number of times needed to work things out. 'Symbolic methods' break things into smaller pieces *and* allow some effective abbreviations. These may not be optimal in easy cases, but start to become very handy in messier situations.

Example: Considering the same example as just above,

$$F(x,y) = f(u(x,y), v(a(y,xy), b(x^2, x^3, y)))$$

find $\frac{\partial}{\partial x} F(x,y,z)$, where now we suppose that

$$u(x,y) = xy$$

$$v(x,y) = x + y$$
$$a(x,y) = e^{xy}$$
$$b(x,y,z) = xy^2z^3$$
$$f(x,y) = x^2 - y^2$$

(Note that in these descriptions of the pieces going into the whole, we have not used symbols as they are actually used in the definition of F. This is totally legal, and in fact follows another convention, which says to always use x as the first input, y as the second, and so on.) It might be wise to rewrite the descriptions of these auxiliary functions using letters that suggest what actually occurs in F: the very first one, $u(x,y)$, is ok, but the others are better changed. Again, the procedure is to **read the function descriptions not in terms of the symbols, but in terms of what role each input plays in the output:**

$$v(a,b) = a + b$$

$$a(y,xy) = y + xy$$
$$b(x^2, x^3, y) = (x^2)(x^3)^2(y)^3 = x^8y^3$$
$$f(u,v) = u^2 - v^2$$

where we use abbreviations $a = a(y,xy)$, $b = b(x^2,x^3,y)$ as before. In particular, planning to invoke the *symbolic* chain rule computation we did in the previous paragraph, we also need

$$f_1(u,v) = \frac{\partial}{\partial u}(u^2 - v^2) = 2u$$

$$f_2(u,v) = \frac{\partial}{\partial u}(u^2 - v^2) = -2v$$

Being more circumspect, since

$$a_2(x,y) = \frac{\partial}{\partial y} e^{xy} = xe^{xy}$$

then

$$a_2(y,xy) = (y)e^{(y)(xy)}$$

by replacing the 'old' x from the previous line by the 'new' y, and replacing the 'old' y by the 'new' xy. Really what we are doing is simply using different first and second inputs. In the same vein, since

$$b_1(x,y,z) = \frac{\partial}{\partial x} xy^2z^3 = y^2z^3$$

we have

$$b_1(x^2,x^3,y) = (x^3)^2(y)^3$$

by replacing the 'old' x (from the previous line) by x^2, replacing 'old' y by x^3, and 'old' z by y (really, just using different inputs to b_2). And in just the same way, since

$$b_2(x,y,z) = \frac{\partial}{\partial y} xy^2z^3 = x(2y)z^3$$

we have

$$b_2(x^2,x^3,y) = (x^2)(2x^3)(y)^3$$

Putting this all together, we have

$$\frac{\partial}{\partial x} F(x,y) = f_1 \cdot u_1 + f_2 \cdot v_1 \cdot [0 + a_2 \cdot y] + f_2 \cdot v_2 \cdot [b_1 \cdot 2x + b_2 \cdot 3x^2 + 0]$$

$$= 2u \cdot \frac{\partial}{\partial x} xy - 2v \cdot 1 \cdot [0 + (y)e^{(y)(xy)} \cdot y] - 2v \cdot 1 \cdot [(x^3)^2(y)^3 \cdot 2x + (x^2)(2x^3)(y)^3 \cdot 3x^2]$$

which we could simplify if we felt like it.

Which approach is best? Well, using a more 'symbolic' approach, as opposed to 'writing it all out', breaks the computation up into smaller pieces, but at the cost of having to *name* the pieces, and do some bookkeeping. Of course, if you don't have explicit formulas for everything, then you *have* to use a more symbolic method since you can't 'write it out'.

In summary, if you have a choice, in simple situations you might just write the function out explicitly and use a single-variable chain rule. At a certain point, though, the expressions you get may not fit on a single line, etc. And the pattern of the computation can be completely unclear. Then it might be wiser, or necessary, to break the problem down into smaller pieces.

What are those 'tree diagrams'? They are just a bookkeeping device to help you not overlook any of the myriad terms that come up in repeated applications of the several-variable chain rule with highly composite functions. It's not *necessary* to use them, but may be soothing.

---

17. Implicit Differentiation

The phrase *implicit differentiation* is misleading: what this is really about is **differentiation of functions implicitly defined**. So it is the *definition* that is indirect, not really the differentiation.

That a function be '*implicitly defined*' means that instead of having a *formula* for it, into which we plug the *inputs* and compute an *output*, the *output* function is tangled up together with the *input* in a manner which does not necessarily allow straightforward evaluation. For example, if all we know of a function f is that

$$f(x)^5 + xf(x) - x^5 = 0$$

it is not clear what kind of formula we could get to express $f(x)$ as something computable in terms of x.

(Nevertheless, for each numerical value of the input x, we could use numerical methods such as Newton-Raphson to find (one or more) numbers $f(x)$ which fit this relation. This falls far short of giving a 'formula', however).

In this context, if we can't get a 'formula' for $f(x)$ in terms of x, it is unreasonable to expect a 'formula' for the derivative $f'(x)$ in terms of x. However, it turns out that we *can* get a formula expressing $f'(x)$ in terms of x **and** $f(x)$. This is certainly not as good as a formula in terms of x alone, but is all that we can generally hope for.

Examples in one variable

Example: Let y be defined as a function of x by the relation $y^3 - xy + x^3 = 1$. Express dy/dx in terms of x and y: It is basically futile to try to 'solve for' y, meaning to express y in terms of x by some formula. Rather, just view the left-hand and right-hand side of this equation as being two functions of x. The right-hand side is a very simple function, visibly equal to 1 all the time. The left-hand side is more complicated in appearance, but is hypothesized to be equal to 1 all the time. This will give an indirect handle on y.

Differentiate both sides with respect to x, keeping in mind that y is a function of x, even though we don't have a formula for it. Thus, we have

$$\frac{d}{dx}(y^3 - xy + x^3) = \frac{d}{dx}1 = 0$$

Using the chain rule,

$$\frac{d}{dx}y^3 = 3y^2 \cdot y'$$

The fact that we can't say what y' is may be vaguely disturbing, but it's ok. Likewise, using the product rule,

$$\frac{d}{dx}(xy) = \frac{d}{dx}x \cdot y + x \cdot \frac{d}{dx}y = y + xy'$$

And, of course,

$$\frac{d}{dx}x^3 = 3x^2$$

Therefore, we have

$$3y^2y' - xy' - y + 3x^2 = 0$$

The slight surprise is that we are able to *solve* for y' in terms of x and y. First regroup according to terms which do or don't involve y':

$$y' \cdot (3y^2 - x) + (-y + 3x^2) = 0$$

Then move the non-y' terms to the other side of the equation and divide by the coefficient of y':

$$y' = \frac{y - 3x^2}{3y^2 - x}$$

Done.

Example: Continuing the previous example: find all possible values that y might have when $x = 1$, and determine all the corresponding values of y':

This question illustrates another quirk of these *implicit* or *indirect* definitions of functions, namely that usually it's not just a single function which gets itself defined, but several at once, which may be tangled together in complicated patterns. So the requirement to *find all possible values* of y corresponding to a single value of the input x is a serious thing.

Fortunately, even though we can't readily solve the equation for y in general, when $x = 1$ it is simpler: when $x = 1$, y must satisfy

$$y^3 - 1 \cdot y + 1^3 = 1$$

which is just

$$y^3 - y = 0$$

This has three solutions, $y = -1, 0, 1$. These give three corresponding values of y':

$$y'|_{(1,-1)} = \frac{y - 3x^2}{3y^2 - x}|_{(1,-1)} = \frac{(-1) - 3(1)^2}{3(-1)^2 - (1)}$$

$$y'|_{(1,0)} = \frac{y - 3x^2}{3y^2 - x}|_{(1,0)} = \frac{(0) - 3(1)^2}{3(0)^2 - (1)}$$

$$y'|_{(1,1)} = \frac{y - 3x^2}{3y^2 - x}|_{(1,1)} = \frac{(1) - 3(1)^2}{3(1)^2 - (1)}$$

Done.

Symbolic perspective on the one-variable case

What really happened in the previous example is that we have a function of *two* variables,

$$f(x, y) = y^3 - xy + x^3$$

and this function was used to require a relation between x and y by

$$f(x, y) = 1$$

which we interpreted as making y a function of x.

In general, if f is a function of two variables, and if y is defined as a function of x by any relation

$$f(x, y) = c$$

then we can do an analogous thing purely symbolically: differentiate both sides as functions of x, using the chain rule to correctly differentiate $f(x, y)$.

$$\frac{d}{dx} f(x, y) = \frac{d}{dx} c = 0$$

Here

$$\frac{d}{dx}f(x,y) = f_1(x,y)\cdot\frac{dx}{dx} + f_2(x,y)\cdot\frac{dx}{dx} = f_1(x,y) + f_2(x,y)\cdot y'$$

So we have

$$f_1(x,y) + f_2(x,y)\cdot y' = 0$$

which is readily solved to give

$$y' = -\frac{f_1(x,y)}{f_2(x,y)}$$

Done.

Notice that this 'formula' really tells the whole story of one-variable implicit differentiation, but that it requires understanding partial differentiation to tell it!

The question of 'being able to tell' whether a relation $f(x,y) = c$ defines y as a function of x or *vice versa* is misguided: this relation could be used either way, to define x as a function of y or y as a function of x. If the context does not make clear what's going on then it's impossible to tell.

Another one-variable example

Example: Let y be defined as a function of x by $f(x^2, x+y) = 3$, where $f(a,b) = ab + b^3$. Find $\partial y/\partial x$.

The way the problem is posed it also invites notational confusion, since the definition of f seems to involve some mysterious entities a and b. In reality, of course, one should read a as 'first input to f', and b as 'second input to f'.

Well, we can just rewrite and write it out: first, replace a by x^2 and b by $x+y$ in the expression for f, to get the *relation*

$$(x^2)(x+y) + (x+y)^3 = 3$$

If we wanted to, we could multiply it out:

$$x^3 + x^2y + x^3 + 3x^2y + 3xy^2 + y^3 = 3$$

Then differentiate both sides as functions of x, where we know that y is to be treated as a function of x even though we don't know what it is by *formula*:

$$3x^2 + (2xy + x^2\frac{\partial y}{\partial x} + 3x^2 + (6xy + 3x^2\frac{\partial y}{\partial x}) + (3y^2 + 6xy\frac{\partial y}{\partial x}) + 3y^2\frac{\partial y}{\partial x} = \frac{\partial 0}{\partial x} = 0$$

Regrouping, this is

$$[6x^2 + 8xy + 3y^2] + \frac{\partial y}{\partial x}[4x^2 + 6xy + 3y^2] = 0$$

which we solve for $\frac{\partial y}{\partial x}$ to get

$$\frac{\partial y}{\partial x} = -\frac{6x^2 + 8xy + 3y^2}{4x^2 + 6xy + 3y^2}$$

Done.

Several-variable examples

Example: Define x as a function of y and z by $x^3 + y^3 + z^3 - xyz = 2$. Find all the possible values that x may have when $y = 1$ and $z = 1$, and find all the corresponding values of $\partial x/\partial z$.

Comment: It is perfectly ok to have a problem in which x is a function of y and z. On the other hand, from the relation alone it is impossible to 'tell' which is a function of what. It is only when someone tells you, or when you have other information from the context, that there is any way to know what is supposed to be a function of what else.

In fact, in real life, figuring out reasonable hypotheses about *'what depends on what'* is sometimes a big part of the problem. While our intuition leads us to believe that 'being a function of' implies some sort of *causality*, it's hard to describe 'causality' in mathematical terms, etc.

In the example, although we cannot readily 'solve' for x to get a formula expressing it in terms of y and z, when $y = 1$ and $z = 1$ the equation becomes more tractable: it is

$$x^3 + 1^3 + 1^3 - x \cdot 1 \cdot 1 = 2$$

which is just $x^3 - x = 0$. This has three roots, $x = -1, 0, 1$.

To get an expression for $\partial x/\partial z$, just differentiate both sides of the equation with respect to z. Since y, z are the 'independent' variables, $\partial y/\partial z = 0$, while $\partial x/\partial z$ is what we want to find. We have

$$\frac{\partial}{\partial z} 2 = 0 = \frac{\partial}{\partial z} x^3 + \frac{\partial}{\partial z} y^3 + \frac{\partial}{\partial z} z^3 - \frac{\partial}{\partial z}(xyz)$$

$$= 3x^2 \frac{\partial x}{\partial z} + 0 + 3z^2 - \frac{\partial x}{\partial z} \cdot yz + xy \cdot 1$$

by using the chain rule, product rule, and the fact that $\partial y/\partial z = 0$ (while $\partial z/\partial z = 1$).

This can be solved for $\partial x/\partial z$ in terms of x, y, z: first regroup

$$\frac{\partial x}{\partial z}(3x^2 - yz) + (3z^2 - xy) = 0$$

which then gives

$$\frac{\partial x}{\partial z} = \frac{xy - 3z^2}{3x^2 - yz}$$

Plugging in the three triples

$$(x, y, z) = (-1, 1, 1) \qquad (x, y, z) = (0, 1, 1) \qquad (x, y, z) = (1, 1, 1)$$

found above, we get values

$$\frac{\partial x}{\partial z}|_{(-1,1,1)} = \frac{xy - 3z^2}{3x^2 - yz}|_{(-1,1,1)} = \frac{(-1)(1) - 3(1)^2}{3(-1)^2 - (1)(1)}$$

$$\frac{\partial x}{\partial z}|_{(0,1,1)} = \frac{xy - 3z^2}{3x^2 - yz}|_{(0,1,1)} = \frac{(0)(1) - 3(1)^2}{3(0)^2 - (1)(1)}$$

$$\frac{\partial x}{\partial z}|_{(1,1,1)} = \frac{xy - 3z^2}{3x^2 - yz}|_{(1,1,1)} = \frac{(1)(1) - 3(1)^2}{3(1)^2 - (1)(1)}$$

Done.

Example: Define x as a function of y, z by $f(x, xy, x + z) = 1$ where $f(a, b, c) = abc + a^3$. Find $\partial x/\partial z$.

First, just write everything out, replacing the inputs a, b, c by $x, xy, x+z$ (in that order) in the definition of f: the relation becomes

$$(x)(xy)(x + z) + x^3 = 1$$

We will differentiate both sides with respect to z, where we consider y as *constant* (since it's the other *independent* variable), and x is some unknown function of z: first multiply out:

$$x^3y + x^2yz + x^3 = 1$$

Now do the differentiation:

$$3x^2 \frac{\partial x}{\partial z} y + 2x \frac{\partial x}{\partial z} yz + x^2 y \cdot 1 + 3x^2 \frac{\partial x}{\partial z} = 0$$

Solving for $\frac{\partial x}{\partial z}$, this gives

$$\frac{\partial x}{\partial z} = -\frac{x^2 y}{3x^2 y + 2xyz + 3x^2}$$

Done.

Symbolic perspective on several-variable examples

Just as the single-variable implicit differentiation had a 'formula' for it, involving partial derivatives of the function used in the defining relation, so there is a 'formula' for the several variable case.

A significant hazard here is that a relation among many variables can be used to define any one of them as a function of the others. That means that there would have to be *many* 'formulas' to cover all the possibilities. Or, on the other hand, it means that there are many more opportunities to *misapply* whatever formulas one knows. My own preference is *not* to try to 'memorize' a *formula*, but to understand the following *procedure*. As often happens, it is possible to give a clearer description of the *procedure* than of the formulas to which the procedure gives rise.

Let $F(x, y, z)$ be a function of three variables. Fix a constant c, and define x as function of y and z by the relation

$$F(x, y, z) = c$$

Express $\partial x / \partial y$ as a function of x, y, z:

Since we are interested in derivatives with respect to y, differentiate both sides of this equality with respect to y, keeping in mind that since y, z are the 'independent' variables, $\partial z / \partial y = 0$. Of course, $\partial y / \partial y = 1$. We have

$$\frac{\partial}{\partial y} F(x, y, z) = \frac{\partial}{\partial y} c = 0$$

The left-hand side is further computed, by the Chain Rule, as

$$\frac{\partial}{\partial y} F(x, y, z) = F_1(x, y, z) \cdot \frac{\partial}{\partial y} x + F_2(x, y, z) \cdot \frac{\partial}{\partial y} y + F_3(x, y, z) \cdot \frac{\partial}{\partial y} z$$

$$= F_1(x, y, z) \cdot \frac{\partial x}{\partial y} + F_2(x, y, z) \cdot 1 + 0$$

Solving for $\partial x / \partial y$ gives

$$\frac{\partial x}{\partial y} = -\frac{F_2(x, y, z)}{F_1(x, y, z)}$$

Done.

Still assuming that x is a function of y and z, a similar computation with the Chain Rule gives

$$\frac{\partial x}{\partial z} = -\frac{F_3(x, y, z)}{F_1(x, y, z)}$$

Or, instead, if we use the *same relation* to define y as a function of x and z, then a similar computation gives

$$\frac{\partial y}{\partial x} = -\frac{F_1(x, y, z)}{F_2(x, y, z)}$$

$$\frac{\partial y}{\partial z} = -\frac{F_3(x, y, z)}{F_2(x, y, z)}$$

Watch out: The notation F_i meaning 'partial derivative with respect to i^{th} input' is significantly better than always writing $\partial F/\partial x$ for the partial derivative with respect to the first input. Especially *here*, there is a terrible ambiguity! Because, after all, y is a function of x, so one could (?) take the viewpoint that F depends upon x not only through the first input, but through the second as well, so $\partial F/\partial x$ would include more than just the partial derivative with respect to the first input?!?!?

Better to avoid this confusion entirely by using better notation.

More examples

Example: We should do a more *symbolic* example of an implicit differentiation, meaning that we don't have 'formulas' describing all the auxiliary functions:

Let y be defined as a function of x, z by $f(x^2, u(x+y+z), z^3) = 1$. Express $\frac{\partial y}{\partial x}$ in terms of x, z, f, and u:

Again, differentiate both sides with respect to x, where we treat z as constant while doing so (since it is the other 'independent' variable) and y is some 'unknown' function of x:

$$f_1 \cdot \frac{\partial x^2}{\partial x} + f_2 \cdot u_1 \cdot \frac{\partial(x+y+z)}{\partial x} + 0 = 0$$

That is,

$$f_1 \cdot 2x + f_2 \cdot u_1 \cdot (1 + \frac{\partial y}{\partial x}) = 0$$

It is easy to solve for $\frac{\partial y}{\partial x}$:

$$\frac{\partial y}{\partial x} = -\frac{f_1 \cdot 2x + f_2 \cdot u_1}{f_2 \cdot u_1}$$

Done.

Example: Define y as a function of x, z by $f(u(x, xy), v(xz, x^2y)) = 0$. Find $\frac{\partial y}{\partial x}$:

Differentiate both sides of the relation with respect to x, treating z (the other independent variable) as constant, while y is some unknown function of x:

$$f_1 \cdot (u_1 \cdot 1 + u_2 \cdot (y + x\frac{\partial y}{\partial x})) + f_2 \cdot (v_1 \cdot z + v_2 \cdot (2xy + x^2\frac{\partial y}{\partial x})) = 0$$

Solving for $\frac{\partial y}{\partial x}$, we get

$$\frac{\partial y}{\partial x} = -\frac{f_1 \cdot u_1 + f_1 \cdot u_2 \cdot y + f_2 \cdot v_1 \cdot z + f_2 \cdot v_2 \cdot 2x}{f_1 \cdot u_2 \cdot x + f_2 \cdot v_2 \cdot x^2}$$

Done.

Comment: Again, in all the examples of implicit differentiation, there is a 'formula' that can be applied: if *one* of $x, y, z, \ldots$ is defined as a function of the *remaining* ones by a relation

$$f(x, y, z, \ldots) = \text{ constant}$$

then there is a 'formula' whose precise form depends upon which is a function of what. And if the function denoted here as 'f' is presented as a *composite*, then the *chain rule* must be used to compute the items below, anyway!

Again, iff z is a function of x, y defined by a relation f(x,y,z)=c, then

$$\frac{\partial z}{\partial x} = -\frac{\partial f/\partial x}{\partial f/\partial z}$$

Or, this might be rewritten as

$$\frac{\partial z}{\partial x} = -\frac{f_1}{f_3}$$

On the other hand, if, instead, x is a function of y and z, then

$$\frac{\partial x}{\partial z} = -\frac{\partial f/\partial z}{\partial f/\partial x} = -\frac{f_3}{f_1}$$

And, further, if (for example) $f(x,y,z) = F(u(x,y,z), v(x,y,z))$, then we can (and probably *must*) express f_1, f_3 in terms of F, u, v and *their* derivatives. Thus, if x is a function of y, z, then

$$\frac{\partial x}{\partial z} = -\frac{f_3}{f_1} = -\frac{F_1 \cdot u_3 + F_2 \cdot v_3}{F_1 \cdot u_1 + F_2 \cdot v_1}$$

18. Critical points, maximization, minimization

There is a several-variables analogue of the single-variable maximization and minimization problem, and the *basic* answer still is to look for maxima and minima where the "derivative" is zero. But since the notion of "derivative" is more complicated in several variables, the precise details are likewise more complicated.

The Basic Goal One fundamental type of mathematical problem is **maximization** or **minimization** of some function, with inputs restricted in some prescribed manner.

That is, given a function and a limitation on 'legal' inputs to it, we are to find the choice(s) of input which give the largest and smallest values of the output, and also see what those largest and smallest values *are*.

It is unreasonable to expect to answer very complicated questions in this direction. In that context, it is lucky that devices from calculus can give at least partial answers to maximization/minimization problems in some relatively simple but also important cases.

Critical Points in One Variable In single-variable calculus, a whole class of problems requires something like *Find the maximum and minimum values of the function f on the interval* $[a, b]$.

The crucial observation is that when f is a reasonable function its derivative f' must be 0 at any point x_o where f achieves a maximum value or minimum value. This is pretty believable if you draw a picture, and it is also *provable*. A point x_o so that $f'(x_o) = 0$ is a **criticial point** of f.

Therefore, to search for maxima and minima we need not search among *all* possible input values in the interval $[a, b]$, but only among *critical points*. It turns out that we must add the *endpoints* to the list, as well.

So, in principle, to find maxima and minima of f on the interval $[a, b]$, we follow the following procedure:

- Solve $f'(x) = 0$ to find critical points
- Discard any critical points not in the interval $[a, b]$
- Add to the list the two endpoints a and b
- Go down the list of critical points together with endpoints, evaluating f at each point on the list
- Compare all the values of f, picking out the maximum and minimum

Possible difficulties? Since we have to evaluate the function for every point on the list, we certainly hope the list is *short*. This is true whether we are computing by hand or by machine, although with different scales.

In the small-scale problems we hope to look at, all the steps but the very first one contain no serious obstacles other than slightly tedious labor. At worst a simple calculator might be needed to evaluate a few things that are hard to do by hand. By contrast, finding the critical points entails *solving an equation* at the very first step. This can be horribly difficult to do by hand, and can require a little finesse to do by machine. The examples just below are designed so that this part is easy.

The point is to appreciate that the only part of this procedure which is supposed to present any hazards to us is the very first step.

Example: Find the maximum and minimum of $f(x) = x^3 - 12x$ on the interval $[-1, 3]$:

First, find the derivative of f, set it equal to zero, and solve, to find the critical points: $f'(x) = 3x^2 - 12$, and $3x^2 - 12 = 0$ simplifies to $x^2 - 4 = 0$, so $x = \pm 2$ are the critical points.

Second, discard the critical point -2 since it does not lie inside the prescribed interval.

Third, add endpoints -1 and 3 to the list, in addition to the remaining critical point 2: the list of points to be considered is $-1, 2, 3$.

Fourth, evaluate f at each of these points:

- $f(-1) = (-1)^3 - 12(-1) = -11$
- $f(2) = (2)^3 - 12(2) = -16$
- $f(3) = (3)^3 - 12(3) = -9$

Last, comparing the values just computed, we see that the maximum is -9, which occurs at $x = 3$, and the minimum is -16, which occurs at 2.

Relative Maxima and Minima There is another type of concept and question that can be addressed by almost the same methods.

A **local maximum** (also called *relative maximum*) of a function f on an interval $[a, b]$ occurs at a point x_o if the value $f(x_o) \geq f(x)$ for all x *near* x_o. That is, we don't care about value of f for possible inputs x far away from x_o. And we won't worry about making the notions of 'near' and 'far' any more precise than this.

Likewise, a **local minimum** (also called *relative minimum*) of a function f on an interval $[a, b]$ occurs at a point x_o if the value $f(x_o) \leq f(x)$ with x *near* x_o.

For contrast, *the* maximum and minimum of f on an interval $[a, b]$ are sometimes called **absolute** maximum and minimum. So the *absolute* maximum and minimum are among the *local* maxima and minima, but there may be local maxima and/or minima which are not absolute. This is easy to suggest by pictures.

The so-called **Second-Derivative Test** is the method of finding local minima and maxima which has an analogue in several variables, so we'll review it now. (The other main method is the so-called First Derivative Test, which has no good analogue in several variables, even though it works better in one-variable calculus). The Second-Derivative Test is executed as follows: to find the local minima and maxima of f on $[a, b]$,

- Find critical points of f
- Remove from the list any critical points not inside $[a, b]$
- For a *critical point* x_o,
- If $f''(x_o) > 0$ then x_o gives a local *minimum*
- If $f''(x_o) < 0$ then x_o gives a local *maximum*
- If $f''(x_o) = 0$ then we reach no conclusion about x_o
- For the *left endpoint* a
- If $f'(a) > 0$ then a gives a local *minimum*
- If $f'(a) < 0$ then a gives a local *maximum*

- For the *right endpoint* b
- If $f'(b) > 0$ then a gives a local *maximum*
- If $f'(b) < 0$ then a gives a local *minimum*

Comments: First, it is certainly true that the first two steps of this procedure are identical to the procedure of the last section for finding (absolute) maximum and minimum. However, other than the fact that both procedures look at critical points, there is not much overlap. Indeed, in general, knowing that something gives a *local* max (or min) gives absolutely no information about whether the point gives an *absolute* max (or min).

Second, the way to remember the two different conclusions reached depending upon $f''(x_o) > 0$ or $f''(x_o) < 0$ is to think of the very simple example $f(x) = x^2$. In this case we know how the thing looks, and that $x = 0$ is a local minimum. At the same time, it is also easy to compute $f''(0) = 2$. Therefore, we conclude that $f''(x_o) > 0$ signifies a local *minimum*.

Third, the two differing inequality conditions for the two endpoints are best kept straight by simple schematic diagrams.

Fourth, the 'no conclusion' case really is just that: we *cannot* conclude that the point is *not* a local min or max, only that we *don't know from anything the Second-Derivative Test tells us.* Yes, there are 'higher-derivative tests' that we *could* pursue to resolve this uncertainty, but we won't.

Example: *Find the relative minima and maxima of* $f(x) = x^3 - 3x + 2$ *on* $[-2, 2]$*:* First, $f'(x) = 3x^2 - 3$, and this has roots $x = \pm 1$, so the critical points are ± 1. They both lie inside the specified interval. The second derivative is $f''(x) = 6x$. Evaluating this at the critical point -1 gives $f''(-1) = -6 < 0$, so -1 gives a local *maximum*. Evaluating the second derivative at the other critical point gives $f''(+1) = 6 > 0$, so $+1$ gives a local *minimum*. Evaluating the first derivative at the left endpoint: $f'(-2) = 3(-2)^2 - 3 = 9 > 0$, so -2 is a local *minimum*. And at the right endpoint: $f'(+2) = 3(+2)^2 - 3 = 9$, so $+2$ is a local *maximum*.

Critical Points in Several Variables For a scalar-valued function f of a vector variable $\vec{x}$, a **critical point** of f is a point $\vec{x}_o$ where

$$\nabla f(\vec{x}_o) = \vec{0}$$

The potential problem in one-variable max/min problems, that of solving the equation $f'(x) = 0$, gets much worse here, since the analogous vector equation $\nabla f(\vec{x}) = \vec{0}$ is really a *system* of equations. Solving such things can quickly become very difficult. In all our examples I will have arranged so that solving the necessary system is *possible*, though maybe *challenging* to varying degrees.

Notice, too, that we haven't yet tried to describe any type of max/min problems. This is because the whole idea is a little messier for a function of several variables than for a function of just one variable.

Example: *Find the critical points of* $f(x, y) = x^2 + y^2$.

The gradient of this f is $\nabla f(x, y) = (2x, 2y)$. To find critical points, solve

$$(2x, 2y) = (0, 0)$$

which is equivalent to the system

$$\begin{cases} 2x &= 0 \\ 2y &= 0 \end{cases}$$

We find that the only critical point is $(0, 0)$.

Example: *Find the critical points of* $f(x, y) = x^2 + 4xy + y^2$.

The gradient of this f is $\nabla f(x, y) = (2x + 4y, 4x + 2y)$. To find critical points, solve

$$(2x + 4y, 4x + 2y) = (0, 0)$$

which is equivalent to the system

$$\begin{cases} 2x + 4y &= 0 \\ 4x + 2y &= 0 \end{cases}$$

A person with some sophistication about *linear* systems of equations would suspect already that $(0, 0)$ was the only solution, because of the 0's on the right-hand side! But since we anticipate solving *non*-linear systems as well, we'll use the method of *Elimination of Variables*. Solving the first equation for x in terms of y, we obtain $x = -2y$. Substituting this into the second equation in place of x, we have $4(-2y) + 2y = 0$, so $y = 0$. Substituting back, $x = 0$ also. So $(0, 0)$ is the only critical point.

Example: *Find the critical points of* $f(x, y) = x^2 + 4xy + y^2 + 2x$.

The gradient of this f is $\nabla f(x, y) = (2x + 4y + 2, 4x + 2y)$. To find critical points, solve

$$(2x + 4y + 2, 4x + 2y) = (0, 0)$$

which is equivalent to the system

$$\begin{cases} 2x + 4y &= -2 \\ 4x + 2y &= 0 \end{cases}$$

Use the method of *Elimination of Variables*. Solving the first equation for x in terms of y, we obtain $x = -2 - 2y$. Substituting this into the second equation in place of x, we have $4(-2 - 2y) + 2y = 0$. This simplifies to $-8 - 4y + 2y = 0$ or $2y = -8$ so $y = -4$. Substituting back, $x = -2 - 2(-4) = 6$ also. So $(6, -4)$ is the only critical point.

Example: *Find the critical points of* $f(x, y) = 2x^3 + 6xy + 3y^2$.

The gradient of this f is $\nabla f(x, y) = (6x^2 + 6y, 6x + 6y)$. To find critical points, solve

$$(6x^2 + 6y, 6x + 6y) = (0, 0)$$

which (dividing out all the 6's) is equivalent to the system

$$\begin{cases} x^2 + y &= 0 \\ x + y &= 0 \end{cases}$$

From the second equation, $x = -y$. Substitute this into the first equation for x, giving $y^2 + y = 0$. This has solutions $y = 0, -1$. Substituting back, we get corresponding values $0, +1$ for x. That is, the critical points are $(0, 0)$ and $(1, -1)$.

Example: *Find the critical points of* $f(x, y) = 3x^4 + 12xy + 4y^3$.

The gradient of this f is $\nabla f(x, y) = (12x^3 + 12y, 12x + 12y^2)$. To find critical points, solve

$$(12x^3 + 12y, 12x + 12y^2) = (0, 0)$$

which (dividing out all the 12's) is equivalent to the system

$$\begin{cases} x^3 + y &= 0 \\ x + y^2 &= 0 \end{cases}$$

From the second equation, $x = -y^2$. Substitute this into the first equation for x, giving $-y^6 + y = 0$. Rearrange this to

$$y(y^5 - 1) = 0$$

Invoking the important (if silly-sounding) principle that *the product of two real numbers is zero only if at least one of the factors is zero*, we see that the solutions of this single equation are obtained from solutions of $y = 0$ *together with* solutions of $y^5 - 1 = 0$. Thus, $y = 0$ is certainly one solution. Further, the equation $y^5 = 1$ has only one *real* solution, namely $y = 1$, since the only real fifth root of 1 is just 1.

Thus, the values of y are $y = 0, 1$. Substituting back into $x = -y^2$, the corresponding values of x are $0, +1$. Thus, the critical points are $(0, 0)$ and $(+1, -1)$.

Example: *Find the critical points of* $f(x, y) = x^4 - 4xy + y^4$*:*

The gradient of this f is $\nabla f(x, y) = (4x^3 - 4y, -4x + 4y^3)$. To find critical points, solve

$$(4x^3 - 4y, -4x + 4y^3) = (0, 0)$$

which (dividing out all the 4's) is equivalent to the system

$$\begin{cases} x^3 - y &= 0 \\ -x + y^3 &= 0 \end{cases}$$

From the second equation, $x = y^3$. Substitute this into the first equation for x, giving $y^9 - y = 0$. Rearrange this to

$$y(y^8 - 1) = 0$$

Invoking the important (if silly-sounding) principle that *the product of two real numbers is zero only if at least one of the factors is zero*, we see that the solutions of this single equation are obtained from solutions of $y = 0$ *together with* solutions of $y^8 - 1 = 0$. Thus, $y = 0$ is certainly one solution. The equation $y^8 = 1$ has exactly two *real* solutions, namely $y = \pm 1$, since these are the only two real 8^{th} roots of 1.

Thus, the values of y are $y = -1, 0, 1$. Substituting back into $x = y^3$, the corresponding values of x are $-1, 0, +1$. Thus, the critical points are $(-1, -1)$, $(0, 0)$ and $(+1, +1)$.

Local Minima and Maxima, Saddle Points

The two-variable version of the Second-Derivative Test requires a more complicated apparatus than the one-variable version. The fact that we stop at the two-variable case certainly is an indicator that higher-dimensional versions are more complicated still.

In two variables, there are at least three different phenomena that can be identified by the Second-Derivative Test. Two of these are familiar already from the one-variable situation: First, a point (x_o, y_o) gives a **local minimum** for the function f if $f(x_o, y_o) \leq f(x, y)$ for all other points (x, y) 'near' (x_o, y_o). A point (x_o, y_o) gives a **local maximum** for the function f if $f(x_o, y_o) \geq f(x, y)$ for all other points (x, y) 'near' (x_o, y_o). The new possibility in two dimensions is that of a **saddle point**, which, as its name suggests, is a point at which the graph of f bends upward along one line, but downward along another line, in the shape of a saddle. A simple function whose graph has a saddle point at $(0, 0)$ is $f(x, y) = xy$.

The general problem we now consider is: Let R be a region in the plane, let $f(x, y)$ be a reasonable function defined on R, and *find all the local minima, local maxima, and saddle points of f in the region R.*

Immediately there is one complication: while in the one-variable case a 'region' was just an interval, and the 'boundary' was just the set of two endpoints, in the two-variable situation the *boundary* of a region is a much more complicated thing. Therefore, analysis of maxima and minima *on the boundary* of the region must be postponed. (The method of *Lagrange multipliers* can be used to address this issue).

That is, we will modify the question to be: *find all the local minima, local maxima, and saddle points of f* **in the interior** *of the region R.* This excludes worry about what happens on the boundary.

The two-variable analogue of 'the second derivative' of a function f is the 2-by-2 *matrix* of all the possible second partial derivatives of f, called the **Hessian** of f.

$$\text{Hessian of } f\text{, evaluated at } (x, y) = \begin{pmatrix} f_{11}(x, y) & f_{12}(x, y) \\ f_{21}(x, y) & f_{22}(x, y) \end{pmatrix}$$

where as usual the subscripts denote partial differentiation with respect to inputs.

The present **two-variable version** of the **Second-Derivative Test** is a method which can answer questions about finding local maxima, local minima, and saddlepoints in the interior of a planar region. This method runs as follows. To find the local maxima, local minima, and saddle points of a function $f(x, y)$ in a region R in the plane,

- Solve $\nabla f(x, y) = (0, 0)$ to find critical points
- Remove from the list any critical points *outside* the region R
- For a critical point (x_o, y_o) *inside* the region R, let $\begin{pmatrix} a & b \\ b & d \end{pmatrix}$ be an abbreviation for the Hessian of f evaluated at (x_o, y_o): that is, for brevity write

$$\begin{pmatrix} a & b \\ b & d \end{pmatrix} = \begin{pmatrix} f_{11}(x, y) & f_{12}(x, y) \\ f_{21}(x, y) & f_{22}(x, y) \end{pmatrix}$$

- If the determinant $ad - b^2$ of the Hessian evaluated at (x_o, y_o) is > 0, and if the diagonal entries a and d are > 0, then (x_o, y_o) gives a local minimum
- If the determinant $ad - b^2$ of the Hessian evaluated at (x_o, y_o) is > 0, and if the diagonal entries a and d are < 0, then (x_o, y_o) gives a local maximum
- If the determinant $ad - b^2$ of the Hessian is *negative*, then (x_o, y_o) gives a saddle point
- If the determinant $ad - b^2$ of the Hessian is 0, then we get no information.

Comments: First, for reasonable functions we know that $\partial^2 f/\partial x\, \partial y = \partial^2 f/\partial y\, \partial x$, which is why the upper-right and lower-left entries in the Hessian are the same.

Second, the expression $ad - b^2$ certainly *is* the determinant of the Hessian evaluated at (x_o, y_o).

Third, if this two-variable version seems disappointingly complicated by comparison to the one-variable version, just reflect a little on how much more complicated the geometry of surfaces is than the geometry of curves. It's fair that it be more complicated.

We'll go through the examples just above whose critical points we already computed.

Example: Find the local maxima, minima, and saddle points of $f(x, y) = x^2 + y^2$ in the region $x^2 + y^2 \le 4$:

We omit the computation (already done, above) which shows that the only critical point is $(0, 0)$. Then

$$\text{Hessian of } f = \begin{pmatrix} f_{11}(x, y) & f_{12}(x, y) \\ f_{21}(x, y) & f_{22}(x, y) \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

That is, the Hessian matrix is *constant.* The determinant of the Hessian is $1 \cdot 1 - 0 \cdot 0 > 0$, so this critical point is a *local minimum.*

Example: *Classify the critical points of* $f(x, y) = x^2 + 4xy + y^2 + 2x$.

We already saw that $(6, -4)$ is the only critical point. The Hessian is

$$\text{Hessian of } f = \begin{pmatrix} f_{11}(x, y) & f_{12}(x, y) \\ f_{21}(x, y) & f_{22}(x, y) \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix}$$

As in the previous example, this is a constant matrix, so evaluation at the point $(6, -4)$ doesn't change it. The determinant is $2 \cdot 2 - 4 \cdot 4 < 0$, so this point is a *saddle point.*

Example: *Classify the critical points of* $f(x,y) = 2x^3 + 6xy + 3y^2$.

We already found that the critical points are $(0,0)$ and $(1,-1)$. The Hessian is

$$\text{Hessian of } f = \begin{pmatrix} f_{11}(x,y) & f_{12}(x,y) \\ f_{21}(x,y) & f_{22}(x,y) \end{pmatrix} = \begin{pmatrix} 12x & 6 \\ 6 & 6 \end{pmatrix}$$

The determinant is $72x - 36$. At the point $(0,0)$, plug in 0 for x, to find that the determinant of the Hessian at that point is $-36 < 0$, so $(0,0)$ is a *saddle point.* At the point $(1,-1)$, put 1 in for x, and the determinant of the Hessian is $72 - 36 > 0$. Further, the diagonal entries at that point are $12 \cdot 1$ and 6, both positive, so $(1,-1)$ is a local *minimum.*

Example: *Classify the critical points of* $f(x,y) = 3x^4 + 12xy + 4y^3$.

We saw that the critical points are $(0,0)$ and $(+1,-1)$. The Hessian is

$$\text{Hessian of } f = \begin{pmatrix} f_{11}(x,y) & f_{12}(x,y) \\ f_{21}(x,y) & f_{22}(x,y) \end{pmatrix} = \begin{pmatrix} 36x^2 & 12 \\ 12 & 24y \end{pmatrix}$$

At $(0,0)$ the value of the Hessian is $\begin{pmatrix} 0 & 12 \\ 12 & 0 \end{pmatrix}$ which has determinant $-12^2 < 0$, so $(0,0)$ is a *saddle point.* At $(+1,-1)$, the Hessian is to $\begin{pmatrix} 36 & 12 \\ 12 & -24 \end{pmatrix}$ which again has negative determinant, so $(+1,-1)$ is another saddle point.

Comment: A person might wonder what this surface looks like.

Example: *Classify the critical points of* $f(x,y) = x^4 - 4xy + y^4$*:*

We saw that the critical points are $(-1,-1)$, $(0,0)$ and $(+1,+1)$. The Hessian is

$$\text{Hessian of } f = \begin{pmatrix} f_{11}(x,y) & f_{12}(x,y) \\ f_{21}(x,y) & f_{22}(x,y) \end{pmatrix} = \begin{pmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{pmatrix}$$

At $(0,0)$ the Hessian is $\begin{pmatrix} 0 & -4 \\ -4 & 0 \end{pmatrix}$ which has negative determinant, so gives a saddle point. At *both* points $\pm(1,1)$, the Hessian is $\begin{pmatrix} 12 & -4 \\ -4 & 12 \end{pmatrix}$ which has positive determinant and positive diagonal entries, so indicates local minima at both $(+1,+1)$ and $(-1,-1)$.