quiz 01.1 Solution

(1) List all functions from the set $A = \{2, 3\}$ to the set $B = \{7, 10, 5\}$.

There are several different notational styles one may use here, though all of them necessarily describe the output for every input. We will pick the most economical one, which by coincidence is the most formal, namely to say that a function is the set of ordered pairs (x, y) where x ranges over all possible inputs and the corresponding y is the output for that input x. In the present context, for each of the two inputs we have three choices for the output, and there are no conditions otherwise. Thus, there are 3×3 such functions altogether. We list them by grouping the 3 with the same output for the input 2 together, in a lexicographic ordering. Thus, the list of all such functions is

$$\{(2,7),(3,7)\}$$

$$\{(2,7),(3,10)\}$$

$$\{(2,7),(3,5)\}$$

$$\{(2,10),(3,7)\}$$

$$\{(2,10),(3,10)\}$$

$$\{(2,10),(3,5)\}$$

$$\{(2,5),(3,7)\}$$

$$\{(2,5),(3,10)\}$$

$$\{(2,5),(3,5)\}$$

(2) How many pairs of disjoint 4-element and 5-element subsets of a 13-element set are there?

Starting from scratch: there are 13 choices for the first element of the first set, $13-1$ for the second, $13-2$ for the third, and so on, so there are $13 \ldots (13-4+1)$ choices for an **ordered** subset of 4 elements. But this style of choosing artificially orders the chosen elements. To take this into account, we divide by 4!, the number of ways to order a set with 4 elements. From the remaining $(13-4)$-element subset, there are $(13-4)$ choices for the first element of the second set, $(13-4)-1$ choices for the second element of the second set, and so on. And we divide by 5! to take into account that our choosing artificially imposes an order where there is none. So for each choice of the first set there are

$$(13-4)(13-4-1)\ldots(13-4-5+1)/5!$$

Thus, altogether there are

$$\frac{13!}{(13-4)!\,4!} \cdot \frac{(13-4)!}{(13-4-5)!\,5!} = \frac{13!}{(13-4-5)!\,4!\,5!}$$

(3) How many unordered triples of disjoint 4-element, 4-element, and 5-element subsets of a 14-element set are there?

Starting from scratch: there are 14 choices for the first element of the first set, $14-1$ for the second, $14-2$ for the third, and so on, so there are $14 \ldots (14-4+1)$ choices for an **ordered** collection of 4 elements. But this style of choosing artificially orders the chosen elements. To take this into account, we divide by 4!, the number of ways to order a set with 4 elements. From the remaining $(14-4)$-element subset, there are $(14-4)$ choices for the first element of the second set, $(14-4)-1$ choices for the second element of the second set, and so on. And we divide by 4! to take into account that our choosing artificially imposes an order where there is none. So for each choice of the second set there are

$$(14-4)(14-4-1)\ldots(14-4-4+1)/4! = \binom{14-4}{4}$$

But, again, we have artificially *ordered* these two sets (as, first and second), so to compensate we must divide by 2!. Thus, altogether there are

$$= \binom{14}{4} = \binom{14-4}{4} \cdot \frac{1}{2}$$

choices of two 4-element sets. From the remaining $(14-4-4)$-element subset, there are $(14-4-4)$ choices for the first element of the third set, $(14-4-4)-1$ choices for the second element of the second set, and so on. And we divide by 5! to take into account that our choosing artificially imposes an order where there is none. So for each choice of the two 4-element subsets there are

$$(14-4-4)(14-4-4-1)\ldots(14-4-4-5+1)/5! = \binom{14-4-4}{5}$$

choices of the 5-element subset. So, in total, there are

$$\binom{14}{4}\binom{14-4}{4} \cdot \frac{1}{2!} \cdot \binom{14-4-4}{5}$$

(4) How many surjective functions are there from a 8-element set to a 2-element set?

Note that we will not *list* them, but only *count* them.

To give things names, let $S = \{s_1, \ldots s_8\}$, $T = \{a, b\}$, and count all the surjections from S to T. For a function $f : S \to T$, let A be the set of elements of S which map to a and let B be the set of elements which map to b. The requirement of surjectivity is that *some* element of S maps to a, and *some* element of T maps to b, which is to say that both A and B have at least one element. Also, note that once A is specified B is specified, since anything not in A is in B, and vice-versa. Thus, to count the number of such surjections f we need only count the number of subsets A of S that have at least one element, and leave at least one element out. This is the number of *all* subsets minus 2, since we must exclude the empty set and the whole set. The number of all subsets of an n-element set is 2^n, so the number of nonempty subsets is $2^n - 2$. Here, this gives $2^8 - 2 = 254$ surjective functions of the sort indicated.