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Cantor-Schroeder-Bernstein Theorem

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/˜garrett/

This is the key result that allows comparison of infinities. Perhaps it is the first serious theorem in set theory after Cantor's diagonalization argument. Apparently Cantor *conjectured* this result, and it was proven independently by F. Bernstein and E. Schröder in the 1890's. This author is of the opinion that the proof given below is the *natural* proof one would find after sufficient experimentation and reflection. [Suppes 1960] gives a somewhat more formal version, and says that this proof is in [Fraenkel 1953], p. 102-103, and is attributed by Fraenkel to J. M. Whitaker. One must mention [Hausdorff 1914] as an influential source which helped to standardize modern usage.

It is noteworthy that there is no invocation of the Axiom of Choice, since one can imagine otherwise.

The argument below is not the most succinct possible, but is intended to lend a greater sense of inevitability to the conclusion than might the shortest possible version.

Theorem: Let A and B be sets, with injections $f : A \to B$ and $g : B \to A$. Then there exists a canonical *bijection* $F : A \to B$.

Proof: Let

$$A_o = \{a \in A : a \notin g(B)\} \qquad B_o = \{b \in B : b \notin f(A)\}$$

The sets

$$A_{2n} = (g \circ f)^n(A_o) \qquad A_{2n+1} = (g \circ f)^n g(B_o)$$

are disjoint. Let A_∞ be the complement in A to the union $\bigcup_n A_n$. Define F by

$$F(a) = \begin{cases} f(a) & (\text{for } a \in A_n,\ n \in 2\mathbf{Z}) \\ g^{-1}(a) & (\text{for } a \in A_n,\ n \in 1 + 2\mathbf{Z}) \\ f(a) & (\text{for } a \in A_\infty) \end{cases}$$

We must verify that this moderately clever apparent definition really gives a well-defined F, and that F is a bijection. For $n \geq 1$, let

$$B_n = f(A_{n-1})$$

and also let $B_\infty = f(A_\infty)$.

The underlying fact is that $A \cup B$ (disjoint union) is *partitioned* into one-sided or two-sided maximal sequences of elements that map to each other under f and g: we have three patterns. First, one may have

$$a_o \xrightarrow{f} b_1 \xrightarrow{g} a_1 \xrightarrow{f} b_2 \xrightarrow{g} a_2 \to \ldots \xrightarrow{f} b_n \xrightarrow{g} a_n \to \ldots$$

beginning with $a_o \in A_o$, all $a_i \in A$ and $b_i \in B$. Second, one may have

$$b_o \xrightarrow{g} a_1 \xrightarrow{f} b_1 \xrightarrow{g} a_2 \xrightarrow{f} b_2 \to \ldots \xrightarrow{g} a_n \xrightarrow{f} b_n \to \ldots$$

with $b_o \in B_o$, and $a_i \in A$ and $b_i \in B$. The third and last possibility is that none of the elements involved is an image of A_o or B_o under any number of iterations of $f \circ g$ or $g \circ f$. Such elements fit into pictures of the form

$$\ldots \xrightarrow{g} a_{-2} \xrightarrow{f} b_{-1} \xrightarrow{g} a_{-1} \xrightarrow{f} b_o \xrightarrow{g} a_o \xrightarrow{f} b_1 \xrightarrow{g} \ldots$$

where $a_i \in A$ and $b_i \in B$. The fundamental point is that any two distinct such sequences of elements are disjoint. And any element certainly lies in such a sequence.

The one-sided sequences of the form

$$a_o \xrightarrow{f} b_1 \xrightarrow{g} a_1 \xrightarrow{f} b_2 \xrightarrow{g} a_2 \to \ldots \xrightarrow{f} b_n \xrightarrow{g} a_n \to \ldots$$

beginning with $a_o \in A_o$, can be broken up to give part of the definition of F by

$$F : a_o \xrightarrow{f} b_1 \quad F : a_1 \xrightarrow{f} b_2 \ldots$$

The one-sided sequences of the form

$$b_o \xrightarrow{g} a_1 \xrightarrow{f} b_1 \xrightarrow{g} a_2 \xrightarrow{f} b_2 \to \ldots \xrightarrow{g} a_n \xrightarrow{f} b_n \to \ldots$$

with $b_o \in B_o$, beginning with $b_o \in B_o$, can be broken up to give another part of the definition of F

$$b_o \xrightarrow{g} a_1 \quad b_1 \xrightarrow{g} a_2 \ldots$$

which is to say

$$F : a_1 \xrightarrow{g^{-1}} b_o \quad F : a_2 \xrightarrow{g^{-1}} b_1 \ldots$$

For a double-sided sequence,

$$\ldots \xrightarrow{g} a_{-2} \xrightarrow{f} b_{-1} \xrightarrow{g} a_{-1} \xrightarrow{f} b_o \xrightarrow{g} a_o \xrightarrow{f} b_1 \xrightarrow{g} \ldots$$

there are two equally simple ways to break it up, and we choose

$$F : a_i \xrightarrow{f} b_{i+1}$$

Since the sequences partition $A \cup B$, and since every element of B (and A) appears, F is surely a bijection from A to B. ///

[Fraenkel 1953] A. Fraenkel, *Abstract Set Theory*, North-Holland, Amsterdam, 1953.

[Hausdorff 1914] F. Hausdorff, *Grundzuge der Mengenlehre*, Leipzig, 1914; reprinted Chelsea, NY, 1949.

[Suppes 1960] P. Suppes, *Axiomatic Set Theory*, Van Nostrand, 1960; reprinted Dover, 1972.
