Discussion 08

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[08.1] Let $T \in \text{Hom}_k(V)$ for a finite-dimensional $k$-vector space $V$, with $k$ a field. Let $W$ be a $T$-stable subspace. Prove that the minimal polynomial of $T$ on $W$ is a divisor of the minimal polynomial of $T$ on $V$. Define a natural action of $T$ on the quotient $V/W$, and prove that the minimal polynomial of $T$ on $V/W$ is a divisor of the minimal polynomial of $T$ on $V$.

Discussion: Let $f(x)$ be the minimal polynomial of $T$ on $V$, and $g(x)$ the minimal polynomial of $T$ on $W$. (We need the $T$-stability of $W$ for this to make sense at all.) Since $f(T) = 0$ on $V$, and since the restriction map

$$\text{End}_k(V) \to \text{End}_k(W)$$

is a ring homomorphism,

$$(\text{restriction of}) f(t) = f(\text{restriction of } T)$$

Thus, $f(T) = 0$ on $W$. That is, by definition of $g(x)$ and the PID-ness of $k[x]$, $f(x)$ is a multiple of $g(x)$, as desired.

Define $T(v + W) = Tv + W$. Since $TW \subset W$, this is well-defined. Note that we cannot assert, and do not need, an equality $TW = W$, but only containment. Let $h(x)$ be the minimal polynomial of $T$ (on $V/W$). Any polynomial $p(T)$ stabilizes $W$, so gives a well-defined map $p(T)$ on $V/W$. Further, since the natural map

$$\text{End}_k(V) \to \text{End}_k(V/W)$$

is a ring homomorphism, we have

$$\overline{p(T)}(v + W) = p(T)(v) + W = p(T)(v + W) + W = p(T)(v + W)$$

Since $f(T) = 0$ on $V$, $f(T) = 0$. By definition of minimal polynomial, $h(x)f(x)$. ///

[08.2] Let $T \in \text{Hom}_k(V)$ for a finite-dimensional $k$-vector space $V$, with $k$ a field. Suppose that $T$ is diagonalizable on $V$. Let $W$ be a $T$-stable subspace of $V$. Show that $T$ is diagonalizable on $W$.

Discussion: Since $T$ is diagonalizable, its minimal polynomial $f(x)$ on $V$ factors into linear factors in $k[x]$ (with zeros exactly the eigenvalues), and no factor is repeated. By the previous example, the minimal polynomial $g(x)$ of $T$ on $W$ divides $f(x)$, so (by unique factorization in $k[x]$) factors into linear factors without repeats. And this implies that $T$ is diagonalizable when restricted to $W$. ///

[08.3] Let $T \in \text{Hom}_k(V)$ for a finite-dimensional $k$-vector space $V$, with $k$ a field. Suppose that $T$ is diagonalizable on $V$, with distinct eigenvalues. Let $S \in \text{Hom}_k(V)$ commute with $T$, in the natural sense that $ST = TS$. Show that $S$ is diagonalizable on $V$.

Discussion: The hypothesis of distinct eigenvalues means that each eigenspace is one-dimensional. We have seen that commuting operators stabilize each other’s eigenspaces. Thus, $S$ stabilizes each one-dimensional $\lambda$-eigenspace $V_\lambda$ for $T$. By the one-dimensionality of $V_\lambda$, $S$ is a scalar $\mu_\lambda$ on $V_\lambda$. That is, the basis of eigenvectors for $T$ is unavoidably a basis of eigenvectors for $S$, too, so $S$ is diagonalizable. ///

[08.4] Let $T \in \text{Hom}_k(V)$ for a finite-dimensional $k$-vector space $V$, with $k$ a field. Suppose that $T$ is diagonalizable on $V$. Show that $k[T]$ contains the projectors to the eigenspaces of $T$.

Discussion: Though it is only implicit, we only want projectors $P$ which commute with $T$. 

1
Since $T$ is diagonalizable, its minimal polynomial $f(x)$ factors into linear factors and has no repeated factors. For each eigenvalue $\lambda$, let $f_\lambda(x) = f(x)/(x - \lambda)$. The hypothesis that no factor is repeated implies that the $gcd$ of all these $f_\lambda(x)$ is 1, so there are polynomials $a_\lambda(x)$ in $k[x]$ such that

$$1 = \sum_\lambda a_\lambda(x) f_\lambda(x)$$

For $\mu \neq \lambda$, the product $f_\lambda(x)f_\mu(x)$ picks up all the linear factors in $f(x)$, so

$$f_\lambda(T)f_\mu(T) = 0$$

Then for each eigenvalue $\mu$

$$(a_\mu(T)f_\mu(T))^2 = (a_\mu(T)f_\mu(T))(1 - \sum_{\lambda \neq \mu} a_\lambda(T) f_\lambda(T)) = (a_\mu(T)f_\mu(T))$$

Thus, $P_\mu = a_\mu(T)f_\mu(T)$ has $P_\mu^2 = P_\mu$. Since $f_\lambda(T)f_\mu(T) = 0$ for $\lambda \neq \mu$, we have $P_\mu P_\lambda = 0$ for $\lambda \neq \mu$. Thus, these are projectors to the eigenspaces of $T$, and, being polynomials in $T$, commute with $T$.

For uniqueness, observe that the diagonalizability of $T$ implies that $V$ is the sum of the $\lambda$-eigenspaces $V_\lambda$ of $T$. We know that any endomorphism (such as a projector) commuting with $T$ stabilizes the eigenspaces of $T$. Thus, given an eigenvalue $\lambda$ of $T$, an endomorphism $P$ commuting with $T$ and such that $P(V) = V_\lambda$ must be 0 on $T$-eigenspaces $V_\mu$ with $\mu \neq \lambda$, since

$$P(V_\mu) \subset V_\mu \cap V_\lambda = 0$$

And when restricted to $V_\lambda$ the operator $P$ is required to be the identity. Since $V$ is the sum of the eigenspaces and $P$ is determined completely on each one, there is only one such $P$ (for each $\lambda$).

[08.5] Let $V$ be a complex vector space with a (positive definite) inner product. Show that $T \in \text{Hom}_k(V)$ cannot be a normal operator if it has any non-trivial Jordan block.

Discussion: The spectral theorem for normal operators asserts, among other things, that normal operators are diagonalizable, in the sense that there is a basis of eigenvectors. We know that this implies that the minimal polynomial has no repeated factors. Presence of a non-trivial Jordan block exactly means that the minimal polynomial does have a repeated factor, so this cannot happen for normal operators.

[08.6] Show that a positive-definite hermitian $n$-by-$n$ matrix $A$ has a unique positive-definite square root $B$ (that is, $B^2 = A$).

Discussion: Even though the question explicitly mentions matrices, it is just as easy to discuss endomorphisms of the vector space $V = \mathbb{C}^n$.

By the spectral theorem, $A$ is diagonalizable, so $V = \mathbb{C}^n$ is the sum of the eigenspaces $V_\lambda$ of $A$. By hermitian-ness these eigenspaces are mutually orthogonal. By positive-definiteness $A$ has positive real eigenvalues $\lambda$, which therefore have real square roots. Define $B$ on each orthogonal summand $V_\lambda$ to be the scalar $\sqrt{\lambda}$. Since these eigenspaces are mutually orthogonal, the operator $B$ so defined really is hermitian, as we now verify. Let $v = \sum_\lambda v_\lambda$ and $w = \sum_\mu w_\mu$ be orthogonal decompositions of two vectors into eigenvectors $v_\lambda$ with eigenvalues $\lambda$ and $w_\mu$ with eigenvalues $\mu$. Then, using the orthogonality of eigenvectors with distinct eigenvalues,

$$\langle Bv, w \rangle = \langle B \sum_\lambda v_\lambda, \sum_\mu w_\mu \rangle = \sum_\lambda \lambda v_\lambda, \sum_\mu w_\mu \rangle = \sum_\lambda \lambda \langle v_\lambda, w_\lambda \rangle$$

$$= \sum_\lambda \langle v_\lambda, \lambda w_\lambda \rangle = \langle \sum_\mu v_\mu, \sum_\lambda \lambda w_\lambda \rangle = \langle v, Bw \rangle$$
Uniqueness is slightly subtler. Since we do not know a priori that two positive-definite square roots $B$ and $C$ of $A$ commute, we cannot immediately say that $B^2 = C^2$ gives $(B + C)(B - C) = 0$, etc. If we could do that, then since $B$ and $C$ are both positive-definite, we could say

$$\langle (B + C)v, v \rangle = \langle Bv, v \rangle + \langle Cv, v \rangle > 0$$

so $B + C$ is positive-definite and, hence invertible. Thus, $B - C = 0$. But we cannot directly do this. We must be more circumspect.

Let $B$ be a positive-definite square root of $A$. Then $B$ commutes with $A$. Thus, $B$ stabilizes each eigenspace of $A$. Since $B$ is diagonalizable on $V$, it is diagonalizable on each eigenspace of $A$ (from an earlier example). Thus, since all eigenvalues of $B$ are positive, and $B^2 = \lambda$ on the $\lambda$-eigenspace $V_\lambda$ of $A$, it must be that $B$ is the scalar $\sqrt{\lambda}$ on $V_\lambda$. That is, $B$ is uniquely determined.

[08.7] Given a square $n$-by-$n$ complex matrix $M$, show that there are unitary matrices $A$ and $B$ such that $AMB$ is diagonal.

Discussion: We prove this for not-necessarily square $M$, with the unitary matrices of appropriate sizes.

This asserted expression

$$M = \text{unitary} \cdot \text{diagonal} \cdot \text{unitary}$$

is called a Cartan decomposition of $M$.

First, if $M$ is (square) invertible, then $T = MM^*$ is self-adjoint and invertible. From an earlier example, the spectral theorem implies that there is a self-adjoint (necessarily invertible) square root $S$ of $T$. Then

$$1 = S^{-1}TS^{-1} = (S^{-1}M)(-SM)^*$$

so $k_1 = S^{-1}M$ is unitary. Let $k_2$ be unitary such that $D = k_2Sk_2^*$ is diagonal, by the spectral theorem. Then

$$M = Sk_1 = (k_2Dk_2^*)k_1 = k_2 \cdot D \cdot (k_2^*k_1)$$

expresses $M$ as

$$M = \text{unitary} \cdot \text{diagonal} \cdot \text{unitary}$$

as desired.

In the case of $m$-by-$n$ (not necessarily invertible) $M$, we want to reduce to the invertible case by showing that there are $m$-by-$m$ unitary $A_1$ and $n$-by-$n$ unitary $B_1$ such that

$$A_1MB_1 = \begin{pmatrix} M' & 0 \\ 0 & 0 \end{pmatrix}$$

where $M'$ is square and invertible. That is, we can (in effect) do column and row reduction with unitary matrices.

Nearly half of the issue is showing that by left (or right) multiplication by a suitable unitary matrix $A$ an arbitrary matrix $M$ may be put in the form

$$AM = \begin{pmatrix} M_{11} & M_{12} \\ 0 & 0 \end{pmatrix}$$

with 0’s below the $r^{th}$ row, where the column space of $M$ has dimension $r$. To this end, let $f_1, \ldots, f_r$ be an orthonormal basis for the column space of $M$, and extend it to an orthonormal basis $f_1, \ldots, f_m$ for the whole $\mathbb{C}^m$. Let $e_1, \ldots, e_m$ be the standard orthonormal basis for $\mathbb{C}^m$. Let $A$ be the linear endomorphism of $\mathbb{C}^m$ defined by $Af_i = e_i$ for all indices $i$. We claim that this $A$ is unitary, and has the desired effect on
That is has the desired effect on $M$ is by design, since any column of the original $M$ will be mapped by $A$ to the span of $e_1, \ldots, e_r$, so will have all 0’s below the $r^{th}$ row. A linear endomorphism is determined exactly by where it sends a basis, so all that needs to be checked is the unitariness, which will result from the orthonormality of the bases, as follows. For $v = \sum_i a_i f_i$ and $w = \sum_i b_i f_i$,

$$\langle Av, Aw \rangle = \langle \sum_i a_i Af_i, \sum_j b_j Af_j \rangle = \langle \sum_i a_i e_i, \sum_j b_j e_j \rangle = \sum_i a_i b_i$$

by orthonormality. And, similarly,

$$\sum_i a_i \bar{b_i} = \langle \sum_i a_i f_i, \sum_j b_j f_j \rangle = \langle v, w \rangle$$

Thus, $\langle Av, Aw \rangle = \langle v, w \rangle$. To be completely scrupulous, we want to see that the latter condition implies that $A^*A = 1$. We have $\langle A^*Av, w \rangle = \langle v, w \rangle$ for all $v$ and $w$. If $A^*A \neq 1$, then for some $v$ we would have $A^*Av \neq v$, and for that $v$ take $w = (A^*A - 1)v$, so

$$\langle (A^*A - 1)v, w \rangle = \langle (A^*A - 1)v, (A^*A - 1)v \rangle > 0$$

contradiction. That is, $A$ is certainly unitary.

If we had had the foresight to prove that row rank is always equal to column rank, then we would know that a combination of the previous left multiplication by unitary and a corresponding right multiplication by unitary would leave us with

$$\begin{pmatrix} M' & 0 \\ 0 & 0 \end{pmatrix}$$

with $M'$ square and invertible, as desired. ///

[08.8] Given a square $n$-by-$n$ complex matrix $M$, show that there is a unitary matrix $A$ such that $AM$ is upper triangular.

**Discussion:** Let $\{e_i\}$ be the standard basis for $\mathbb{C}^n$. To say that a matrix is upper triangular is to assert that (with left multiplication of column vectors) each of the maximal family of nested subspaces (called a maximal flag)

$$V_0 = 0 \subset V_1 = \mathbb{C}e_1 \subset \mathbb{C}e_1 + \mathbb{C}e_2 \subset \ldots \subset \mathbb{C}e_1 + \ldots + \mathbb{C}e_{n-1} \subset V_n = \mathbb{C}^n$$

is stabilized by the matrix. Of course

$$MV_0 \subset MV_1 \subset MV_2 \subset \ldots \subset MV_{n-1} \subset V_n$$

is another maximal flag. Let $f_{i+1}$ be a unit-length vector in the orthogonal complement to $MV_i$ inside $MV_{i+1}$ Thus, these $f_i$ are an orthonormal basis for $V$, and, in fact, $f_1, \ldots, f_t$ is an orthonormal basis for $MV_i$. Then let $A$ be the unitary endomorphism such that $Af_i = e_i$. (In an earlier example and in class we checked that, indeed, a linear map which sends one orthonormal basis to another is unitary.) Then

$$AMV_i = V_i$$

so $AM$ is upper-triangular. ///

[08.9] Let $Z$ be an $m$-by-$n$ complex matrix. Let $Z^*$ be its conjugate-transpose. Show that

$$\det(1_m - ZZ^*) = \det(1_n - Z^*Z)$$
Discussion: Write $Z$ in the (rectangular) Cartan decomposition

$$Z = ADB$$

with $A$ and $B$ unitary and $D$ is $m$-by-$n$ of the form

$$
D = \begin{pmatrix}
d_1 & & \\
& d_2 & \\
& & \ddots & \\
& & & d_r \\
& & & & 0 \\
& & & & & \ddots
\end{pmatrix}
$$

where the diagonal $d_i$ are the only non-zero entries. We grant ourselves that $\det(xy) = \det(x) \cdot \det(y)$ for square matrices $x, y$ of the same size. Then

$$
\det(1_m - ZZ^*) = \det(A) \cdot \det(1_m - DD^*) = \prod_i (1 - d_i d_i^*)
$$

Similarly,

$$
\det(1_n - Z^*Z) = \det(B) \cdot \det(1_n - D^*D) = \prod_i (1 - d_i d_i^*)
$$

which is the same as the first computation.///

[08.10] Give an example of two commuting diagonalizable operators $S, T$ on a 4-dimensional vectorspace $V$ over a field $k$ such that each operator has exactly two eigenvalues, and the eigenspaces are two-dimensional, but/and the intersection of any $S$-eigenspace with any $T$-eigenspace is just 1-dimensional. Explain why this does not contradict results about simultaneous eigenvectors

Discussion: Let

$$S = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

act on $V = k^4$ (column vectors). Then the standard basis elements $e_1, e_2, e_3, e_4$ are the joint eigenvectors for $S$ and $T$, with four different ordered pairs of eigenvalues.

The too-naive assertion that for commuting endomorphisms $S, T$ the eigenvectors of $S$ are eigenvectors for $T$ is contradicted by this example, since any vector $ae_1 + be_2$ (with $a, b \in k$) is an eigenvector for $S$, but not for $T$ unless $ab = 0$.

But, of course, this is perfectly fine, since there is still a basis consisting of joint eigenvectors. The point is that it is unwise to hastily choose a basis of eigenvectors for one operator, hoping or presuming that they’ll be eigenvectors for the other operator. For example, $e_1 + e_2, e_2, e_3, e_4$ is a basis consisting of $S$-eigenvectors, but the first of these is not an eigenvector for $T$. ///

[08.11] Let $T$ be a diagonalizable operator on a finite-dimensional vector space $V$ over a field $k$. Suppose that some $T$-eigenspace is not one-dimensional. Exhibit a diagonalizable endomorphism $S$ of $V$ commuting with $T$ not lying in $k[T]$. 

5
Discussion: Let $P \in k[T]$ be the projector to the $\lambda$-eigenspace $V_\lambda$ of $T$, with $\dim V_\lambda > 1$. Let $A$ be a non-scalar endomorphism of $V_\lambda$. Such $A$ exists exactly because $\dim V_\lambda > 1$. Then take $S = A \circ P$. This commutes with $T$, because for any $v \in V$

\[ ST(v) = (AP)T(v) = A(TPv) = A(\lambda \cdot P) = \lambda \cdot AP = T(APv) \]

since $APv \in V_\lambda$. Yet, given a polynomial $f(x)$, take $v \in V_\lambda$, and compute

\[ f(T)(v) = f(\lambda) \cdot v \]

so $f(T)$ is a scalar operator on $V_\lambda$. Thus, $S$ is not of the form $f(T)$. ///

[08.12] Let $\lambda_1, \ldots, \lambda_n$ be distinct elements of a field $k$. Let $\mu_1, \ldots, \mu_n$ be arbitrary elements of $k$. Show that there is a unique polynomial $f(x)$ in $k[x]$ of degree $\leq n - 1$ such that $f(\lambda_i) = \mu_i$ for all $i$.

Discussion: [This is Lagrange interpolation again.]

[08.13] Let $T$ be a diagonalizable operator on a finite-dimensional vector space $V$ over a field $k$. Suppose that all the eigenspaces are one-dimensional. Prove that any endomorphism commuting with $T$ is in $k[T]$.

Discussion: We know that an endomorphism $S$ commuting with $T$ stabilizes the eigenspaces of $T$. Since each eigenspace $V_\lambda$ is just one-dimensional, $S$ acts by a scalar $\mu_\lambda$ on $V_\lambda$. Let $f(x)$ be the minimal polynomial of $T$, and $f_\lambda(x) = f(x)/(x - \lambda)$. These polynomials have $gcd$ 1, so there are polynomials $a_\lambda(x)$ such that

\[ 1 = \sum \lambda a_\lambda(x) \cdot f_\lambda(x) \]

As observed earlier,

\[ \text{id}_V = \sum \lambda a_\lambda(T) \cdot f_\lambda(T) \]

and $f_\lambda(T)$ is 0 on $V_\mu$ for $\mu \neq \lambda$. Further, $P_\lambda = a_\lambda(T)f_\lambda(T)$ is in $k[T]$ and is the projector to $V_\lambda$. Then

\[ S = \sum \lambda \mu_\lambda \cdot P_\lambda \in k[T] \]

as claimed. ///

[08.14] Let $S, T$ be commuting diagonalizable endomorphisms of a finite-dimensional vector space $V$ over a field $k$. Suppose that there is a basis $\{v_1, \ldots, v_n\}$ of simultaneous eigenvectors such that for $i \neq j$ the two vectors $v_i$ and $v_j$ either have different eigenvalues for $S$ or have different eigenvalues for $T$. Show that there is a single diagonalizable operator $R$ on $V$ such that $k[S, T] = k[R]$.

Discussion: Let $P_\lambda \in k[S]$ be the projector to the $\lambda$-eigenspace of $S$, and $Q_\mu \in k[T]$ the projector to the $\mu$-eigenspace of $T$. Since $S$ and $T$ commute, $P_\lambda Q_\mu = Q_\mu P_\lambda$ is a projector commuting with both $S$ and $T$. We claim that $P_\lambda Q_\mu$ is the projector to the joint eigenspace where $S$ is $\lambda$ and $T$ is $\mu$. Certainly $Q_\mu$ maps the whole space to the $\mu$ eigenspace for $T$. Since $P_\lambda$ commutes with $T$, it stabilizes this eigenspace, so $(P_\lambda Q_\mu)(V)$ is contained in the $\mu$-eigenspace of $T$. Symmetrically, it is contained in the $\lambda$ eigenspace for $S$, so is contained in the joint eigenspace. On the other hand, for a joint eigenvector $v$ with $Sv = \lambda v$ and $Tv = \mu v$, we have

\[ (P_\lambda Q_\mu)(v) = P_\lambda(Q_\mu v) = P_\lambda(v) = v \]

In the form the question is asked, let $v_i$ have eigenvalue $\lambda_i$ for $S$ and $\mu_i$ for $T$. Then $E_i = P_\lambda Q_{\mu_i}$ is a family of mutually orthogonal projectors (meaning that $E_i E_j = 0$ unless $i = j$, in which case it is $E_i$), whose sum is the identity endomorphism on $V$. 6
Now assume that there are at least \( n \) distinct elements \( \alpha_1, \ldots, \alpha_n \) in the field \( k \), and let
\[
R = \sum_i \alpha_i \cdot E_i \in k[S,T]
\]
By arrangement \( R \) is diagonalizable and has one-dimensional eigenspaces. Since \( S \) and \( T \) commute with \( R \), by an earlier example both \( S \) and \( T \) are in \( k[R] \). Thus, \( k[R] = k[S,T] \).

\[08.15]\] Give an example of a diagonalizable operator \( T \) on a 2-dimensional complex vector space \( V \) (with hermitian inner product \( \langle , \rangle \)) with eigenvectors \( v, w \) such that application of the Gram-Schmidt process does not yield two orthonormal eigenvectors.

**Discussion:** Let \( V = \mathbb{C}^2 \) (column vectors) with the usual hermitian inner product, and let \( T = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \).
It has eigenvectors \( v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) (eigenvalue 1) and \( v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \) (eigenvalue 2). These are not orthogonal to each other. If we apply Gram-Schmidt, instead of \( v_2 \) we have
\[
v'_2 = v_2 - \frac{\langle v_2, v_1 \rangle}{\langle v_1, v_1 \rangle} \cdot v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\]
The latter vector is not an eigenvector.

\[08.16]\] Let \( S \) be a hermitian operator on a finite-dimensional complex vector space \( V \) with hermitian inner product \( \langle , \rangle \). Let \( W \) be a \( S \)-stable subspace of \( V \). Show that \( S \) is hermitian on \( W \).

**Discussion:** Let \( S_0 \) be the restriction of \( S \) to \( W \). If we show that for all \( w, w' \in W \)
\[
\langle S_0 w, w' \rangle = \langle w, S_0 w' \rangle
\]
then by the uniqueness of adjoints \( S_0^* = S_0 \). Indeed, because
\[
\langle Sw, w' \rangle = \langle w, Sw' \rangle
\]
for \( w, w' \) in the whole space \( V \), the identity certainly holds for \( w, w' \in W \).

\[0.1\] Remark: In a similar vein, one can directly show more generally that, for a normal endomorphism \( T \) on \( V \) stabilizing a subspace \( W \), the restriction of \( T^* \) to \( W \) is the adjoint of the restriction of \( T \) to \( W \).

\[08.17]\] Let \( S, T \) be commuting hermitian operators on a finite-dimensional complex vector space \( V \) with hermitian inner product \( \langle , \rangle \). Show that there is an orthonormal basis for \( V \) consisting of simultaneous eigenvectors for both \( S \) and \( T \).

**Discussion:** First, the Spectral Theorem for \( S \) says that \( V \) is the orthogonal direct sum of the eigenspaces \( V_\lambda \) for \( S \). We know that \( T \) stabilizes each such eigenspace. From the previous example, the restriction of \( T \) to each \( V_\lambda \) is still hermitian, so on each \( V_\lambda \) there is an orthonormal basis \( \{ e^{\lambda}_i \} \) consisting of eigenvectors for \( T \) (and \( \lambda \)-eigenvectors for \( S \)). Then, since the different eigenspaces \( V_\lambda \) are mutually orthogonal, the aggregate \( \{ e^{\lambda}_i \} \) is an orthonormal basis for all of \( V \).

\[08.18]\] Let \( k \) be a field, and \( V \) a finite-dimensional \( k \) vectorspace. Let \( \Lambda \) be a subset of the dual space \( V^* \), with \( |\Lambda| < \dim V \). Show that the **homogeneous system of equations**
\[
\lambda(v) = 0 \quad (\text{for all } \lambda \in \Lambda)
\]
has a non-trivial (that is, non-zero) solution \( v \in V \) (meeting all these conditions).
Discussion: The dimension of the span $W$ of $\Lambda$ is strictly less than $\dim V^*$, which we’ve proven is $\dim V^* = \dim V$. We may also identify $V \approx V^{**}$ via the natural isomorphism. With that identification, we may say that the set of solutions is $W^\perp$, and

$$\dim(W^\perp) + \dim W = \dim V^* = \dim V$$

Thus, $\dim W^\perp > 0$, so there are non-zero solutions.

08.19 Let $k$ be a field, and $V$ a finite-dimensional $k$ vectorspace. Let $\Lambda$ be a linearly independent subset of the dual space $V^*$. Let $\lambda \rightarrow a_\lambda$ be a set map $\Lambda \rightarrow k$. Show that an inhomogeneous system of equations

$$\lambda(v) = a_\lambda \ (\text{for all } \lambda \in \Lambda)$$

has a solution $v \in V$ (meeting all these conditions).

Discussion: Let $m = |\Lambda|$, $\Lambda = \{\lambda_1, \ldots, \lambda_m\}$. One way to use the linear independence of the functionals in $\Lambda$ is to extend $\Lambda$ to a basis $\lambda_1, \ldots, \lambda_n$ for $V^*$, and let $e_1, \ldots, e_n \in V^{**}$ be the corresponding dual basis for $V^{**}$. Then let $v_1, \ldots, v_n$ be the images of the $e_i$ in $V$ under the natural isomorphism $V^{**} \approx V$. (This achieves the effect of making the $\lambda_i$ be a dual basis to the $v_i$. We had only literally proven that one can go from a basis of a vector space to a dual basis of its dual, and not the reverse.) Then

$$v = \sum_{1 \leq i \leq m} a_{\lambda_i} \cdot v_i$$

is a solution to the indicated set of equations, since

$$\lambda_j(v) = \sum_{1 \leq i \leq m} a_{\lambda_i} \cdot \lambda_j(v_i) = a_{\lambda_j}$$

for all indices $j \leq m$.

08.20 Let $T$ be a $k$-linear endomorphism of a finite-dimensional $k$-vectorspace $V$. For an eigenvalue $\lambda$ of $T$, let $V_\lambda$ be the generalized $\lambda$-eigenspace

$$V_\lambda = \{v \in V : (T - \lambda)^n v = 0 \text{ for some } 1 \leq n \in \mathbb{Z}\}$$

Show that the projector $P$ of $V$ to $V_\lambda$ (commuting with $T$) lies inside $k[T]$.

Discussion: First we do this assuming that the minimal polynomial of $T$ factors into linear factors in $k[x]$.

Let $f(x)$ be the minimal polynomial of $T$, and let $f_\lambda(x) = f(x)/(x - \lambda)^r$ where $(x - \lambda)^r$ is the precise power of $(x - \lambda)$ dividing $f(x)$. Then the collection of all $f_\lambda(x)$’s has $gcd 1$, so there are $a_\lambda(x) \in k[x]$ such that

$$1 = \sum_{\lambda} a_\lambda(x) f_\lambda(x)$$

We claim that $E_\lambda = a_\lambda(T)f_\lambda(T)$ is a projector to the generalized $\lambda$-eigenspace $V_\lambda$. Indeed, for $v \in V_\lambda$,

$$v = 1 v = \sum_{\mu} a_\mu(T)f_\mu(T) \cdot v = \sum_{\mu} a_\mu(T)f_\mu(T) \cdot v = a_\lambda(T)f_\lambda(T) \cdot v$$

since $(x - \lambda)^r$ divides $f_\mu(x)$ for $\mu \neq \lambda$, and $(T - \lambda)^r v = 0$. That is, it acts as the identity on $V_\lambda$. And

$$(T - \lambda)^r \circ E_\lambda = a_\lambda(T) f(T) = 0 \in \text{End}_k(V)$$
so the image of \( E_\lambda \) is inside \( V_\lambda \). Since \( E_\lambda \) is the identity on \( V_\lambda \), it must be that the image of \( E_\lambda \) is exactly \( V_\lambda \). For \( \mu \neq \lambda \), since \( f(x)|f_\mu(x)f_\lambda(x) \), \( E_\mu E_\lambda = 0 \), so these idempotents are mutually orthogonal. Then

\[
(a_\lambda(T)f_\lambda(T))^2 = (a_\lambda(T)f_\lambda(T)) \cdot (1 - \sum_{\mu \neq \lambda} a_\mu(T)f_\mu(T)) = a_\lambda(T)f_\lambda(T) - 0
\]

That is, \( E_\lambda^2 = E_\lambda \), so \( E_\lambda \) is a projector to \( V_\lambda \).

The mutual orthogonality of the idempotents will yield the fact that \( V \) is the direct sum of all the generalized eigenspaces of \( T \). Indeed, for any \( v \in V \),

\[
v = 1 \cdot v = (\sum_\lambda E_\lambda) v = \sum_\lambda (E_\lambda v)
\]

and \( E_\lambda v \in V_\lambda \). Thus,

\[
\sum_\lambda V_\lambda = V
\]

To check that the sum is (unsurprisingly) direct, let \( v_\lambda \in V_\lambda \), and suppose

\[
\sum_\lambda v_\lambda = 0
\]

Then \( v_\lambda = E_\lambda v_\lambda \), for all \( \lambda \). Then apply \( E_\mu \) and invoke the orthogonality of the idempotents to obtain

\[
v_\mu = 0
\]

This proves the linear independence, and that the sum is direct.

To prove uniqueness of a projector \( E \) to \( V_\lambda \) commuting with \( T \), note that any operator \( S \) commuting with \( T \) necessarily stabilizes all the generalized eigenspaces of \( T \), since for \( v \in V_\mu \),

\[
(T - \lambda)^e S v = S (T - \lambda)^e v = S \cdot 0 = 0
\]

Thus, \( E \) stabilizes all the \( V_\mu \)'s. Since \( V \) is the direct sum of the \( V_\mu \) and \( E \) maps \( V \) to \( V_\lambda \), it must be that \( E \) is 0 on \( V_\mu \) for \( \mu \neq \lambda \). Thus,

\[
E = 1 \cdot E_\lambda + \sum_{\mu \neq \lambda} 0 \cdot E_\mu = E_\lambda
\]

That is, there is just one projector to \( V_\lambda \) that also commutes with \( T \). This finishes things under the assumption that \( f(x) \) factors into linear factors in \( k[x] \).

The more general situation is similar. More generally, for a monic irreducible \( P(x) \) in \( k[x] \) dividing \( f(x) \), with \( P(x)^e \) the precise power of \( P(x) \) dividing \( f(x) \), let

\[
f_P(x) = f(x)/P(x)^e
\]

Then these \( f_P \) have \( \text{gcd} \ 1 \), so there are \( a_P(x) \) in \( k[x] \) such that

\[
1 = \sum_P a_P(x) \cdot f_P(x)
\]

Let \( E_P = a_P(T)f_P(T) \). Since \( f(x) \) divides \( f_P(x) \cdot f_Q(x) \) for distinct irreducibles \( P,Q \), we have \( E_P \circ E_Q = 0 \) for \( P \neq Q \). And

\[
E_P^2 = E_P(1 - \sum_{Q \neq P} E_Q) = E_P
\]
so (as in the simpler version) the $E_P$’s are mutually orthogonal idempotents. And, similarly, $V$ is the direct sum of the subspaces

$$V_P = E_P \cdot V$$

We can also characterize $V_P$ as the kernel of $P^n(T)$ on $V$, where $P^n(x)$ is the power of $P(x)$ dividing $f(x)$. If $P(x) = (x - \lambda)$, then $V_P$ is the generalized $\lambda$-eigenspace, and $E_P$ is the projector to it.

If $E$ were another projector to $V_\lambda$ commuting with $T$, then $E$ stabilizes $V_P$ for all irreducibles $P$ dividing the minimal polynomial $f$ of $T$, and $E$ is 0 on $V_Q$ for $Q \neq (x - \lambda)$, and $E$ is 1 on $V_\lambda$. That is,

$$E = 1 \cdot E_{x-\lambda} + \sum_{Q \neq x-\lambda} 0 \cdot E_Q = E_P$$

This proves the uniqueness even in general. //

**08.21** Let $T$ be a matrix in Jordan normal form with entries in a field $k$. Let $T_{ss}$ be the matrix obtained by converting all the off-diagonal 1’s to 0’s, making $T$ diagonal. Show that $T_{ss}$ is in $k[T]$.

**Discussion:** This implicitly demands that the minimal polynomial of $T$ factors into linear factors in $k[x]$.

Continuing as in the previous example, let $E_\lambda \in k[T]$ be the projector to the generalized $\lambda$-eigenspace $V_\lambda$, and keep in mind that we have shown that $V$ is the direct sum of the generalized eigenspaces, equivalent, that $\sum \lambda E_\lambda = 1$. By definition, the operator $T_{ss}$ is the scalar operator $\lambda$ on $V_\lambda$. Then

$$T_{ss} = \sum \lambda \cdot E_\lambda \in k[T]$$

since (from the previous example) each $E_\lambda$ is in $k[T]$.

**08.22** Let $M = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$ be a matrix in a block decomposition, where $A$ is $m$-by-$m$ and $D$ is $n$-by-$n$.

Show that

$$\det M = \det A \cdot \det D$$

**Discussion:** One way to prove this is to use the formula for the determinant of an $N$-by-$N$ matrix

$$\det C = \sum_{\pi \in S_N} \sigma(\pi) a_{\pi(1),1} \cdots a_{\pi(N),N}$$

where $c_{ij}$ is the $(i,j)^{th}$ entry of $C$, $\pi$ is summed over the symmetric group $S_N$, and $\sigma$ is the sign homomorphism. Applying this to the matrix $M$,

$$\det M = \sum_{\pi \in S_{m+n}} \sigma(\pi) M_{\pi(1),1} \cdots M_{\pi(m+n),m+n}$$

where $M_{ij}$ is the $(i,j)^{th}$ entry. Since the entries $M_{ij}$ with $1 \leq j \leq m$ and $m < i \leq m+n$ are all 0, we should only sum over $\pi$ with the property that

$$\pi(j) \leq m \quad \text{for} \quad 1 \leq j \leq m$$

That is, $\pi$ stabilizes the subset $\{1, \ldots, m\}$ of the indexing set. Since $\pi$ is a bijection of the index set, necessarily such $\pi$ stabilizes $\{m+1, m+2, \ldots, m+n\}$, also. Conversely, each pair $(\pi_1, \pi_2)$ of permutation $\pi_1$ of the first $m$ indices and $\pi_2$ of the last $n$ indices gives a permutation of the whole set of indices.

Let $X$ be the set of the permutations $\pi \in S_{m+n}$ that stabilize $\{1, \ldots, m\}$. For each $\pi \in X$, let $\pi_1$ be the restriction of $\pi$ to $\{1, \ldots, m\}$, and let $\pi_2$ be the restriction to $\{m+1, \ldots, m+n\}$. And, in fact, if we plan
to index the entries of the block $D$ in the usual way, we’d better be able to think of $\pi_2$ as a permutation of \{1, \ldots, n\}, also. Note that $\sigma(\pi) = \sigma(\pi_1)\sigma(\pi_2)$. Then

$$
det M = \sum_{\pi \in \mathcal{X}} \sigma(\pi) M_{\pi(1),1} \cdots M_{\pi(m+n),m+n}$$

$$= \sum_{\pi \in \mathcal{X}} \sigma(\pi) (M_{\pi(1),1} \cdots M_{\pi(m),m}) \cdot (M_{\pi(m+1),m+1} \cdots M_{\pi(m+n),m+n})$$

$$= \left( \sum_{\pi_1 \in \mathcal{S}_n} \sigma(\pi_1) M_{\pi_1(1),1} \cdots M_{\pi_1(m),m} \right) \cdot \left( \sum_{\pi_2 \in \mathcal{S}_n} \sigma(\pi_2) (M_{\pi_2(m+1),m+1} \cdots M_{\pi_2(m+n),m+n}) \right)$$

$$= \left( \sum_{\pi_1 \in \mathcal{S}_n} \sigma(\pi_1) A_{\pi_1(1),1} \cdots A_{\pi_1(m),m} \right) \cdot \left( \sum_{\pi_2 \in \mathcal{S}_n} \sigma(\pi_2) D_{\pi_2(1),1} \cdots D_{\pi_2(n),n} \right) = det A \cdot det D$$

where in the last part we have mapped \{m + 1, \ldots, m + n\} bijectively by $\ell \to \ell - m$. //

[08.23] The so-called Kronecker product\(^1\) of an $m$-by-$m$ matrix $A$ and an $n$-by-$n$ matrix $B$ is

$$A \otimes B = \begin{pmatrix} A_{11} \cdot B & A_{12} \cdot B & \cdots & A_{1m} \cdot B \\ A_{21} \cdot B & A_{22} \cdot B & \cdots & A_{2m} \cdot B \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} \cdot B & A_{m2} \cdot B & \cdots & A_{mn} \cdot B \end{pmatrix}$$

where, as it may appear, the matrix $B$ is inserted as $n$-by-$n$ blocks, multiplied by the respective entries $A_{ij}$ of $A$. Prove that

$$det(A \otimes B) = (det A)^n \cdot (det B)^m$$

at least for $m = n = 2$.

**Discussion:** If no entry of the first row of $A$ is non-zero, then both sides of the desired equality are 0, and we’re done. So suppose some entry $A_{1i}$ of the first row of $A$ is non-zero. If $i \neq 1$, then for $\ell = 1, \ldots, n$ interchange the $\ell^{\text{th}}$ and $(i - 1)n + \ell^{\text{th}}$ columns of $A \otimes B$, thus multiplying the determinant by $(-1)^n$. This is compatible with the formula, so we’ll assume that $A_{11} \neq 0$ to do an induction on $m$.

We will manipulate $n$-by-$n$ blocks of scalar multiples of $B$ rather than actual scalars.

Thus, assuming that $A_{11} \neq 0$, we want to subtract multiples of the left column of $n$-by-$n$ blocks from the blocks further to the right, to make the top $n$-by-$n$ blocks all 0 (apart from the leftmost block, $A_{11}B$). In terms of manipulations of columns, for $\ell = 1, \ldots, n$ and $j = 2, 3, \ldots, m$ subtract $A_{1j}/A_{11}$ times the $\ell^{\text{th}}$ column of $A \otimes B$ from the $((j - 1)n + \ell)^{\text{th}}$ column. Since for $1 \leq \ell \leq n$ the $\ell^{\text{th}}$ column of $A \otimes B$ is $A_{11}$ times the $\ell^{\text{th}}$ column of $B$, and the $((j - 1)n + \ell)^{\text{th}}$ column of $A \otimes B$ is $A_{1j}$ times the $\ell^{\text{th}}$ column of $B$, this has the desired effect of killing off the $n$-by-$n$ blocks along the top of $A \otimes B$ except for the leftmost block. And the $(i, j)^{\text{th}}$ $n$-by-$n$ block of $A \otimes B$ has become $(A_{ij} - A_{1j}A_{11}/A_{11}) \cdot B$. Let

$$A'_{ij} = A_{ij} - A_{1j}A_{11}/A_{11}$$

and let $D$ be the $(m - 1)$-by-$(m - 1)$ matrix with $(i, j)^{\text{th}}$ entry $D_{ij} = A'_{(i-1),(j-1)}$. Thus, the manipulation so far gives

$$det(A \otimes B) = det \begin{pmatrix} A_{11}B & 0 \\ * & D \otimes B \end{pmatrix}$$

\(^1\) As we will see shortly, this is really a tensor product, and we will treat this question more sensibly.
By the previous example (or its transpose)
\[
\det\left( \begin{array}{cc} A_{11}B & 0 \\ D & \otimes B \end{array} \right) = \det(A_{11}B) \cdot \det(D \otimes B) = A_{11}^n \det(B) \cdot \det(D \otimes B)
\]
by the multilinearity of \( \det \).

And, at the same time subtracting \( A_{1j} / A_{11} \) times the first column of \( A \) from the \( j \)th column of \( A \) for \( 2 \leq j \leq m \) does not change the determinant, and the new matrix is
\[
\left( \begin{array}{cc} A_{11} & 0 \\ \ast & D \end{array} \right)
\]

Also by the previous example,
\[
\det A = \det\left( \begin{array}{cc} A_{11} & 0 \\ \ast & D \end{array} \right) = A_{11} \cdot \det D
\]

Thus, putting the two computations together,
\[
\det(A \otimes B) = A_{11}^n \det B \cdot \det(D \otimes B) = A_{11}^n \det B \cdot (\det D)^n (\det B)^{m-1} = (A_{11} \det D)^n \det B \cdot (\det B)^{m-1} = (\det A)^n (\det B)^m
\]
as claimed.

Another approach to this is to observe that, in these terms, \( A \otimes B \) is
\[
\left( \begin{array}{cccc} A_{11} & 0 & \cdots & 0 \\ 0 & A_{11} & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & A_{11} & 0 \\ A_{m1} & 0 & \cdots & 0 \\ 0 & A_{m1} & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & A_{m1} & 0 \\ 0 & \cdots & \cdots & A_{mm} \end{array} \right)
\]
where there are \( m \) copies of \( B \) on the diagonal. By suitable permutations of rows and columns (with an interchange of rows for each interchange of columns, thus giving no net change of sign), the matrix containing the \( A_{ij} \)'s becomes
\[
\left( \begin{array}{cccc} A & 0 & \cdots & 0 \\ 0 & A & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & A \end{array} \right)
\]
with \( n \) copies of \( A \) on the diagonal. Thus,
\[
\det(A \otimes B) = \det\left( \begin{array}{cccc} A & 0 & \cdots & 0 \\ 0 & A & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & A \end{array} \right) \cdot \det\left( \begin{array}{cccc} B & 0 & \cdots & 0 \\ 0 & B & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & B \end{array} \right) = (\det A)^n \cdot (\det B)^m
\]
This might be more attractive than the first argument, depending on one’s tastes.
For distinct primes $p, q$, compute

$$\mathbb{Z}/p \otimes_{\mathbb{Z}/pq} \mathbb{Z}/q$$

where for a divisor $d$ of an integer $n$ the abelian group $\mathbb{Z}/d$ is given the $\mathbb{Z}/n$-module structure by

$$(r + n\mathbb{Z}) \cdot (x + d\mathbb{Z}) = rx + d\mathbb{Z}$$

**Discussion:** We claim that this tensor product is 0. To prove this, it suffices to prove that every $m \otimes n$ (the image of $m \times n$ in the tensor product) is 0, since we have shown that these monomial tensors always generate the tensor product.

Since $p$ and $q$ are relatively prime, there exist integers $a, b$ such that $1 = ap + bq$. Then for all $m \in \mathbb{Z}/p$ and $n \in \mathbb{Z}/q$,

$$m \otimes n = 1 \cdot (m \otimes n) = (ap + bq)(m \otimes n) = a(pm \otimes n) + b(m \otimes qn) = a \cdot 0 + b \cdot 0 = 0$$

An auxiliary point is to recognize that, indeed, $\mathbb{Z}/p$ and $\mathbb{Z}/q$ really are $\mathbb{Z}/pq$-modules, and that the equation $1 = ap + bq$ still does make sense inside $\mathbb{Z}/pq$.

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**08.25** Compute $\mathbb{Z}/n \otimes_{\mathbb{Z}} \mathbb{Q}$ with $0 < n \in \mathbb{Z}$.

**Discussion:** We claim that the tensor product is 0. It suffices to show that every $m \otimes n$ is 0, since these monomials generate the tensor product. For any $x \in \mathbb{Z}/n$ and $y \in \mathbb{Q}$,

$$x \otimes y = x \otimes (n \cdot \frac{y}{n}) = (nx) \otimes \frac{y}{n} = 0 \otimes \frac{y}{n} = 0$$

as claimed.

---

**08.26** Compute $\mathbb{Z}/n \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}$ with $0 < n \in \mathbb{Z}$.

**Discussion:** We claim that the tensor product is 0. It suffices to show that every $m \otimes n$ is 0, since these monomials generate the tensor product. For any $x \in \mathbb{Z}/n$ and $y \in \mathbb{Q}/\mathbb{Z}$,

$$x \otimes y = x \otimes (n \cdot \frac{y}{n}) = (nx) \otimes \frac{y}{n} = 0 \otimes \frac{y}{n} = 0$$

as claimed.

---

**08.27** Compute $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Q}/\mathbb{Z})$ for $0 < n \in \mathbb{Z}$.

**Discussion:** Let $q : \mathbb{Z} \to \mathbb{Z}/n$ be the natural quotient map. Given $\varphi \in \text{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Q}/\mathbb{Z})$, the composite $\varphi \circ q$ is a $\mathbb{Z}$-homomorphism from the free $\mathbb{Z}$-module $\mathbb{Z}$ (on one generator 1) to $\mathbb{Q}/\mathbb{Z}$. A homomorphism $\Phi \in \text{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$ is completely determined by the image of 1 (since $\Phi(\ell) = \Phi(\ell \cdot 1) = \ell \cdot \Phi(1)$), and since $\mathbb{Z}$ is free this image can be anything in the target $\mathbb{Q}/\mathbb{Z}$.

Such a homomorphism $\Phi \in \text{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$ factors through $\mathbb{Z}/n$ if and only if $\Phi(n) = 0$, that is, $n \cdot \Phi(1) = 0$. A complete list of representatives for equivalence classes in $\mathbb{Q}/\mathbb{Z}$ annihilated by $n$ is $0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \ldots, \frac{n-1}{n}$. Thus, $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Q}/\mathbb{Z})$ is in bijection with this set, by

$$\varphi_{i/n}(x + n\mathbb{Z}) = ix/n + \mathbb{Z}$$

In fact, we see that $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Q}/\mathbb{Z})$ is an abelian group isomorphic to $\mathbb{Z}/n$, with

$$\varphi_{1/n}(x + n\mathbb{Z}) = x/n + \mathbb{Z}$$
as a generator.

\[08.28\] Compute \(\mathbb{Q} \otimes \mathbb{Z} \mathbb{Q}\).

**Discussion:** We claim that this tensor product is isomorphic to \(\mathbb{Q}\), via the \(\mathbb{Z}\)-linear map \(\beta\) induced from the \(\mathbb{Z}\)-bilinear map \(B : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}\) given by

\[B : x \times y \to xy\]

First, observe that the monomials \(x \otimes 1\) generate the tensor product. Indeed, given \(a/b \in \mathbb{Q}\) (with \(a, b\) integers, \(b \neq 0\)) we have

\[x \otimes \frac{a}{b} = (\frac{x}{b} \cdot b) \otimes \frac{a}{b} = \frac{x}{b} \otimes (b \cdot \frac{a}{b}) = \frac{x}{b} \otimes a = \frac{x}{b} \otimes a \cdot 1 = (a \cdot \frac{x}{b}) \otimes 1 = \frac{ax}{b} \otimes 1\]

proving the claim. Further, any finite \(\mathbb{Z}\)-linear combination of such elements can be rewritten as a single one: letting \(n_i \in \mathbb{Z}\) and \(x_i \in \mathbb{Q}\), we have

\[\sum_i n_i \cdot (x_i \otimes 1) = (\sum_i n_i x_i) \otimes 1\]

This gives an outer bound for the size of the tensor product. Now we need an inner bound, to know that there is no further collapsing in the tensor product.

From the defining property of the tensor product there exists a (unique) \(\mathbb{Z}\)-linear map from the tensor product to \(\mathbb{Q}\), through which \(B\) factors. We have \(B(x, 1) = x\), so the induced \(\mathbb{Z}\)-linear map \(\beta\) is a bijection on \(\{x \otimes 1 : x \in \mathbb{Q}\}\), so it is an isomorphism.

\[08.29\] Compute \((\mathbb{Q}/\mathbb{Z}) \otimes \mathbb{Z} \mathbb{Q}\).

**Discussion:** We claim that the tensor product is 0. It suffices to show that every \(m \otimes n\) is 0, since these monomials generate the tensor product. Given \(x \in \mathbb{Q}/\mathbb{Z}\), let \(0 < n \in \mathbb{Z}\) such that \(nx = 0\). For any \(y \in \mathbb{Q}\),

\[x \otimes y = x \otimes (n \cdot \frac{y}{n}) = (nx) \otimes \frac{y}{n} = 0 \otimes \frac{y}{n} = 0\]

as claimed.

\[08.30\] Compute \((\mathbb{Q}/\mathbb{Z}) \otimes \mathbb{Z} (\mathbb{Q}/\mathbb{Z})\).

**Discussion:** We claim that the tensor product is 0. It suffices to show that every \(m \otimes n\) is 0, since these monomials generate the tensor product. Given \(x \in \mathbb{Q}/\mathbb{Z}\), let \(0 < n \in \mathbb{Z}\) such that \(nx = 0\). For any \(y \in \mathbb{Q}/\mathbb{Z}\),

\[x \otimes y = x \otimes (n \cdot \frac{y}{n}) = (nx) \otimes \frac{y}{n} = 0 \otimes \frac{y}{n} = 0\]

as claimed. Note that we do not claim that \(\mathbb{Q}/\mathbb{Z}\) is a \(\mathbb{Q}\)-module (it is not!), but only that for given \(y \in \mathbb{Q}/\mathbb{Z}\) there is another element \(z \in \mathbb{Q}/\mathbb{Z}\) such that \(nz = y\). That is, \(\mathbb{Q}/\mathbb{Z}\) is a **divisible** \(\mathbb{Z}\)-module.

\[08.31\] Prove that for a subring \(R\) of a commutative ring \(S\), with \(1_R = 1_S\), polynomial rings \(R[x]\) behave well with respect to tensor products, namely that (as rings)

\[R[x] \otimes_R S \approx S[x]\]

**Discussion:** Given an \(R\)-algebra homomorphism \(\varphi : R \to A\) and \(a \in A\), let \(\Phi : R[x] \to A\) be the unique \(R\)-algebra homomorphism \(R[x] \to A\) which is \(\varphi\) on \(R\) and such that \(\varphi(x) = a\). In particular, this works
for a an \( S \)-algebra and \( \varphi \) the restriction to \( R \) of an \( S \)-algebra homomorphism \( \varphi : S \to A \). By the defining property of the tensor product, the bilinear map \( B : R[x] \times S \to A \) given by

\[
B(P(x) \times s) = s \cdot \Phi(P(x))
\]
gives a unique \( R \)-module map \( \beta : R[x] \otimes_R S \to A \). Thus, the tensor product has most of the properties necessary for it to be the free \( S \)-algebra on one generator \( x \otimes 1 \).

\[0.2\] Remark: However, we might be concerned about verification that each such \( \beta \) is an \( S \)-algebra map, rather than just an \( R \)-module map. We can certainly write an expression that appears to describe the multiplication, by

\[
(P(x) \otimes s) \cdot (Q(x) \otimes t) = P(x)Q(x) \otimes st
\]
for polynomials \( P, Q \) and \( s, t \in S \). If it is well-defined, then it is visibly associative, distributive, etc., as required.

\[0.3\] Remark: The \( S \)-module structure itself is more straightforward: for any \( R \)-module \( M \) the tensor product \( M \otimes_R S \) has a natural \( S \)-module structure given by

\[
s \cdot (m \otimes t) = m \otimes st
\]
for \( s, t \in S \) and \( m \in M \). But one could object that this structure is chosen at random. To argue that this is a good way to convert \( M \) into an \( S \)-module, we claim that for any other \( S \)-module \( N \) we have a natural isomorphism of abelian groups

\[
\text{Hom}_S(M \otimes_R S, N) \cong \text{Hom}_R(M, N)
\]
(where on the right-hand side we simply forget that \( N \) had more structure than that of \( R \)-module). The map is given by

\[
\Phi \mapsto \varphi_\Phi \quad \text{where} \quad \varphi_\Phi(m) = \Phi(m \otimes 1)
\]
and has inverse

\[
\varphi \longmapsto \Phi_\varphi \quad \text{where} \quad \Phi_\varphi(m \otimes s) = s \cdot \varphi(m)
\]
One might further carefully verify that these two maps are inverses.

\[0.4\] Remark: The definition of the tensor product does give an \( R \)-linear map

\[
\beta : R[x] \otimes_R S \to S[x]
\]
associated to the \( R \)-bilinear \( B : R[x] \times S \to S[x] \) by

\[
B(P(x) \otimes s) = s \cdot P(x)
\]
for \( P(x) \in R[x] \) and \( s \in S \). But it does not seem trivial to prove that this gives an isomorphism. Instead, it may be better to use the universal mapping property of a free algebra. In any case, there would still remain the issue of proving that the induced maps are \( S \)-algebra maps.

\[08.32\] Let \( K \) be a field extension of a field \( k \). Let \( f(x) \in k[x] \). Show that

\[
k[x]/f \otimes_k K \cong K[x]/f
\]
where the indicated quotients are by the ideals generated by \( f \) in \( k[x] \) and \( K[x] \), respectively.

**Discussion:** Upon reflection, one should realize that we want to prove isomorphism as \( K[x] \)-modules. Thus, we implicitly use the facts that \( k[x]/f \) is a \( k[x] \)-module, that \( k[x] \otimes_k K \cong K[x] \) as \( K \)-algebras, and that \( M \otimes_k K \) gives a \( k[x] \)-module \( M \) a \( K[x] \)-module structure by

\[
(\sum_i s_i x^i) \cdot (m \otimes 1) = \sum_i (x^i \cdot m) \otimes s_i
\]
The map
\[ k[x] \otimes_k K \approx_{\text{ring}} K[x] / f \]
has kernel (in \( K[x] \)) exactly of multiples \( Q(x) \cdot f(x) \) of \( f(x) \) by polynomials \( Q(x) = \sum_i s_i x^i \) in \( K[x] \). The inverse image of such a polynomial via the isomorphism is
\[ \sum_i x^i f(x) \otimes s_i \]
Let \( I \) be the ideal generated in \( k[x] \) by \( f \), and \( \tilde{I} \) the ideal generated by \( f \) in \( K[x] \). The \( k \)-bilinear map
\[ k[x]/f \times K \to K[x]/f \]
by
\[ B : (P(x) + I) \times s \to s \cdot P(x) + \tilde{I} \]
gives a map \( \beta : k[x]/f \otimes_k K \to K[x]/f \). The map \( \beta \) is surjective, since
\[ \beta(\sum_i (x^i + I) \otimes s_i) = \sum_i s_i x^i + \tilde{I} \]
hits every polynomial \( \sum_i s_i x^i \mod \tilde{I} \). On the other hand, if
\[ \beta(\sum_i (x^i + I) \otimes s_i) \in \tilde{I} \]
then \( \sum_i s_i x^i = F(x) \cdot f(x) \) for some \( F(x) \in K[x] \). Let \( F(x) = \sum_j t_j x^j \). With \( f(x) = \sum_\ell c_\ell x^\ell \), we have
\[ s_i = \sum_{j+\ell=i} t_j c_\ell \]
Then, using \( k \)-linearity,
\[ \sum_i (x^i + I) \otimes s_i = \sum_i \left( x^i + I \otimes \left( \sum_{j+\ell=i} t_j c_\ell \right) \right) = \sum_{j,\ell} \left( x^{j+\ell} + I \otimes t_j c_\ell \right) \]
\[ = \sum_{j,\ell} (c_\ell x^{j+\ell} + I \otimes t_j) = \sum_\ell (\sum_j c_\ell x^{j+\ell} + I) \otimes t_j = \sum_j (f(x)x^j + I) \otimes t_j = \sum_j 0 = 0 \]
So the map is a bijection, so is an isomorphism.

[08.33] Let \( K \) be a field extension of a field \( k \). Let \( V \) be a finite-dimensional \( k \)-vectorspace. Show that \( V \otimes_k K \) is a good definition of the extension of scalars of \( V \) from \( k \) to \( K \), in the sense that for any \( K \)-vectorspace \( W \)
\[ \text{Hom}_K(V \otimes_k K, W) \approx \text{Hom}_k(V, W) \]
where in \( \text{Hom}_k(V, W) \) we forget that \( W \) was a \( K \)-vectorspace, and only think of it as a \( k \)-vectorspace.

Discussion: This is a special case of a general phenomenon regarding extension of scalars. For any \( k \)-vectorspace \( V \) the tensor product \( V \otimes_k K \) has a natural \( K \)-module structure given by
\[ s \cdot (v \otimes t) = v \otimes st \]
for \( s, t \in K \) and \( v \in V \). To argue that this is a *good* way to convert \( k \)-vectorspaces \( V \) into \( K \)-vectorspaces, claim that for any other \( K \)-module \( W \) have a natural isomorphism of abelian groups

\[
\text{Hom}_K(V \otimes_k K, W) \approx \text{Hom}_K(V, W)
\]

On the right-hand side we *forget* that \( W \) had more structure than that of \( k \)-vectorspace. The map is

\[
\Phi \to \varphi_\Phi \quad \text{where} \quad \varphi_\Phi(v) = \Phi(v \otimes 1)
\]

and has inverse

\[
\Phi_\varphi \leftarrow \varphi \quad \text{where} \quad \Phi_\varphi(v \otimes s) = s \cdot \varphi(v)
\]

To verify that these are mutual inverses, compute

\[
\varphi_{\Phi_\varphi}(v) = \Phi_\varphi(v \otimes 1) = 1 \cdot \varphi(v) = \varphi(v)
\]

and

\[
\Phi_{\varphi_\Phi}(v \otimes 1) = 1 \cdot \varphi_{\Phi}(v) = \Phi(v \otimes 1)
\]

which proves that the maps are inverses. ///

**Remark:** In fact, the two spaces of homomorphisms in the isomorphism can be given natural structures of \( K \)-vectorspaces, and the isomorphism just constructed can be verified to respect this additional structure. The \( K \)-vectorspace structure on the left is clear, namely

\[
(s \cdot \Phi)(m \otimes t) = \Phi(m \otimes st) = s \cdot \Phi(m \otimes t)
\]

The structure on the right is

\[
(s \cdot \varphi)(m) = s \cdot \varphi(m)
\]

The latter has only the one presentation, since only \( W \) is a \( K \)-vectorspace.

**Discussion:** Let \( M \) and \( N \) be free \( R \)-modules, where \( R \) is a commutative ring with identity. Prove that \( M \otimes_R N \) is free and

\[
\text{rank} M \otimes_R N = \text{rank} M \cdot \text{rank} N
\]

To verify this, let \( \varphi : X \times Y \to Z \) be a set map. For each fixed \( y \in Y \), the map \( x \to \varphi(x, y) \) factors through a unique \( R \)-module map \( B_y : M \to Z \). For each \( m \in M \), the map \( y \to B_y(m) \) gives rise to a unique \( R \)-linear map \( n \to B(m, n) \) such that

\[
B(m, j(y)) = B_y(m)
\]

The linearity in the second argument assures that we still have the linearity in the first, since for \( n = \sum_t r_t j(y_t) \) we have

\[
B(m, n) = B(m, \sum_t r_t j(y_t)) = \sum_t r_t B_y(m)
\]

which is a linear combination of linear functions. Thus, there is a unique map to \( Z \) induced on the tensor product, showing that the tensor product with set map \( i \times j : X \times Y \to M \otimes_R N \) is free. ///

**Discussion:** Let \( M \) be a free \( R \)-module of rank \( r \), where \( R \) is a commutative ring with identity. Let \( S \) be a commutative ring with identity containing \( R \), such that \( 1_R = 1_S \). Prove that as an \( S \) module \( M \otimes_R S \) is free of rank \( r \).
Discussion: We prove a bit more. First, instead of simply an inclusion \( R \subset S \), we can consider any ring homomorphism \( \psi: R \to S \) such that \( \psi(1_R) = 1_S \).

Also, we can consider arbitrary sets of generators, and give more details. Let \( M \) be free on generators \( i: X \to M \), where \( X \) is a set. Let \( \tau: M \times S \to M \otimes_R S \) be the canonical map. We claim that \( M \otimes_R S \) is free on \( j: X \to M \otimes_R S \) defined by

\[
j(x) = \tau(i(x) \times 1_S)
\]

Given an \( S \)-module \( N \), we can be a little forgetful and consider \( N \) as an \( R \)-module via \( \psi \), by \( r \cdot n = \psi(r)n \).

Then, given a set map \( \varphi: X \to N \), since \( M \) is free, there is a unique \( R \)-module map \( \Phi: M \to N \) such that \( \varphi = \Phi \circ i \). That is, the diagram

\[
\begin{array}{ccc}
M & \xrightarrow{i} & X \\
\downarrow{\Phi} & & \varphi \\
N & \xrightarrow{\varphi} & N
\end{array}
\]

commutes. Then the map

\[
\psi: M \times S \to N
\]

by

\[
\psi(m \times s) = s \cdot \Phi(m)
\]

induces (by the defining property of \( M \otimes_R S \)) a unique \( \Psi: M \otimes_R S \to N \) making a commutative diagram

\[
\begin{array}{ccc}
M \otimes_R S & \xrightarrow{\tau} & M \times S \\
\downarrow{i \times \text{inc}} & & \downarrow{\psi} \\
X \times \{1_S\} & \xrightarrow{\varphi} & N
\end{array}
\]

where \( \text{inc} \) is the inclusion map \( \{1_S\} \to S \), and where \( t: X \to X \times \{1_S\} \) by \( x \to x \times 1_S \). Thus, \( M \otimes_R S \) is free on the composite \( j: X \to M \otimes_R S \) defined to be the composite of the vertical maps in that last diagram. This argument does not depend upon finiteness of the generating set. ///

[08.36] For finite-dimensional vectorspaces \( V, W \) over a field \( k \), prove that there is a natural isomorphism

\[
(V \otimes_k W)^* \approx V^* \otimes W^*
\]

where \( X^* = \text{Hom}_k(X, k) \) for a \( k \)-vectorspace \( X \).

Discussion: For finite-dimensional \( V \) and \( W \), since \( V \otimes_k W \) is free on the cartesian product of the generators for \( V \) and \( W \), the dimensions of the two sides match. We make an isomorphism from right to left. Create a bilinear map

\[
V^* \times W^* \to (V \otimes_k W)^*
\]

as follows. Given \( \lambda \in V^* \) and \( \mu \in W^* \), as usual make \( \Lambda_{\lambda, \mu} \in (V \otimes_k W)^* \) from the bilinear map

\[
B_{\lambda, \mu}: V \times W \to k
\]

defined by

\[
B_{\lambda, \mu}(v, w) = \lambda(v) \cdot \mu(w)
\]
This induces a unique functional $\Lambda_{\lambda,\mu}$ on the tensor product. This induces a unique linear map

$$V^* \otimes W^* \rightarrow (V \otimes_k W)^*$$

as desired.

Since everything is finite-dimensional, bijectivity will follow from injectivity. Let $e_1, \ldots, e_m$ be a basis for $V$, $f_1, \ldots, f_n$ a basis for $W$, and $\lambda_1, \ldots, \lambda_m$ and $\mu_1, \ldots, \mu_n$ corresponding dual bases. We have shown that a basis of a tensor product of free modules is free on the cartesian product of the generators. Suppose that $\sum_{ij} c_{ij} \lambda_i \otimes \mu_j$ gives the 0 functional on $V \otimes W$, for some scalars $c_{ij}$. Then, for every pair of indices $s, t$, the function is 0 on $e_s \otimes f_t$. That is,

$$0 = \sum_{ij} c_{ij} \lambda_i(e_s) \lambda_j(f_t) = c_{st}$$

Thus, all constants $c_{ij}$ are 0, proving that the map is injective. Then a dimension count proves the isomorphism. 

For a finite-dimensional $k$-vectorspace $V$, prove that the bilinear map

$$B : V \times V^* \rightarrow \text{End}_k(V)$$

by

$$B(v \times \lambda)(x) = \lambda(x) \cdot v$$

gives an isomorphism $V \otimes_k V^* \rightarrow \text{End}_k(V)$. Further, show that the composition of endomorphisms is the same as the map induced from the map on

$$(V \otimes V^*) \times (V \otimes V^*) \rightarrow V \otimes V^*$$

given by

$$(v \otimes \lambda) \times (w \otimes \mu) \rightarrow \lambda(w)v \otimes \mu$$

**Discussion:** The bilinear map $v \times \lambda \rightarrow T_{v,\lambda}$ given by

$$T_{v,\lambda}(w) = \lambda(w) \cdot v$$

induces a *unique* linear map $j : V \otimes V^* \rightarrow \text{End}_k(V)$.

To prove that $j$ is injective, we may use the fact that a basis of a tensor product of free modules is free on the cartesian product of the generators. Thus, let $e_1, \ldots, e_n$ be a basis for $V$, and $\lambda_1, \ldots, \lambda_n$ a dual basis for $V^*$. Suppose that

$$\sum_{i,j=1}^n c_{ij} e_i \otimes \lambda_j \rightarrow 0\text{End}_k(V)$$

That is, for every $e_\ell$,

$$\sum_{ij} c_{ij} \lambda_j(e_\ell)e_i = 0 \in V$$

This is

$$\sum_{i} c_{ij} e_i = 0 \quad (\text{for all } j)$$

Since the $e_i$s are linearly independent, all the $c_{ij}$s are 0. Thus, the map $j$ is injective. Then counting $k$-dimensions shows that this $j$ is a $k$-linear isomorphism.

Composition of endomorphisms is a bilinear map

$$\text{End}_k(V) \times \text{End}_k(V) \rightarrow \text{End}_k(V)$$
Denote by 
\[ c : (v \otimes \lambda) \times (w \otimes \mu) \to \lambda(w)v \otimes \mu \]
the allegedly corresponding map on the tensor products. The induced map on \((V \otimes V^*) \otimes (V \otimes V^*)\) is an example of a contraction map on tensors. We want to show that the diagram

\[
\begin{array}{ccc}
\text{End}_k(V) \times \text{End}_k(V) & \to & \text{End}_k(V) \\
\circ & j \times j & \circ \\
(V \otimes_k V^*) \times (V \otimes_k V^*) & \to & V \otimes_k V^*
\end{array}
\]

commutes. It suffices to check this starting with \((v \otimes \lambda) \times (w \otimes \mu)\) in the lower left corner. Let \(x \in V\). Going up, then to the right, we obtain the endomorphism which maps \(x\) to 
\[
j(v \otimes \lambda) \circ j(w \otimes \mu) \quad (x) = j(v \otimes \lambda)(j(w \otimes \mu)(x)) = j(v \otimes \lambda)(\mu(x) w) \\
= \mu(x) j(v \otimes \lambda)(w) = \mu(x) \lambda(w) v
\]
Going the other way around, to the right then up, we obtain the endomorphism which maps \(x\) to 
\[
j(c((v \otimes \lambda) \times (w \otimes \mu))) \quad (x) = j(\lambda(w)(v \otimes \mu)) \quad (x) = \lambda(w) \mu(x) v
\]
These two outcomes are the same.

For the isomorphism of the previous problem, show that the linear map 
\[
\text{tr} : \text{End}_k(V) \to k
\]
is the linear map 
\[
V \otimes V^* \to k
\]
induced by the bilinear map \(v \times \lambda \to \lambda(v)\).

**Discussion:** Note that the induced map 
\[
V \otimes_k V^* \to k \quad \text{by} \quad v \otimes \lambda \to \lambda(v)
\]
is another contraction map on tensors. Part of the issue is to compare the coordinate-bound trace with the induced (contraction) map \(t(v \otimes \lambda) = \lambda(v)\) determined uniquely from the bilinear map \(v \times \lambda \to \lambda(v)\). To this end, let \(e_1, \ldots, e_n\) be a basis for \(V\), with dual basis \(\lambda_1, \ldots, \lambda_n\). The corresponding matrix coefficients 
\[ T_{ij} \in k \] of a \(k\)-linear endomorphism \(T\) of \(V\) are 
\[
T_{ij} = \lambda_i(T e_j)
\]
(Always there is the worry about interchange of the indices.) Thus, in these coordinates, 
\[
\text{tr}T = \sum_i \lambda_i(T e_i)
\]
Let \(T = j(e_s \otimes \lambda_t)\). Then, since \(\lambda_i(e_i) = 0\) unless \(i = t\),
\[
\text{tr}T = \sum_i \lambda_i(T e_i) = \sum_i \lambda_i(j(e_s \otimes \lambda_t)e_i) = \sum_i \lambda_i(\lambda_t(e_i) \cdot e_s) = \lambda_t(\lambda_s(e_t) \cdot e_s) = \begin{cases} 
1 & (s = t) \\
0 & (s \neq t)
\end{cases}
\]
On the other hand,

\[ t(e_s \otimes \lambda_t) = \lambda_t(e_s) = \begin{cases} 1 & (s = t) \\ 0 & (s \neq t) \end{cases} \]

Thus, these two \( k \)-linear functionals agree on the monomials, which span, they are equal. //

\[08.39\] Prove that \( \text{tr}(AB) = \text{tr}(BA) \) for two endomorphisms of a finite-dimensional vector space \( V \) over a field \( k \), with trace defined as just above.

**Discussion:** Since the maps

\[ \text{End}_k(V) \times \text{End}_k(V) \rightarrow k \]

by

\[ A \times B \rightarrow \text{tr}(AB) \quad \text{and/or} \quad A \times B \rightarrow \text{tr}(BA) \]

are bilinear, it suffices to prove the equality on (images of) monomials \( v \otimes \lambda \), since these span the endomorphisms over \( k \). Previous examples have converted the issue to one concerning \( V \otimes k V^* \). (We have already shown that the isomorphism \( V \otimes_k V^* \approx \text{End}_k(V) \) is converts a contraction map on tensors to composition of endomorphisms, and that the trace on tensors defined as another contraction corresponds to the trace of matrices.) Let \( \text{tr} \) now denote the contraction-map trace on tensors, and (temporarily) write

\[(v \otimes \lambda) \circ (w \otimes \mu) = \lambda(w) v \otimes \mu\]

for the contraction-map composition of endomorphisms. Thus, we must show that

\[ \text{tr} \ (v \otimes \lambda) \circ (w \otimes \mu) = \text{tr} \ (w \otimes \mu) \circ (v \otimes \lambda) \]

The left-hand side is

\[ \text{tr} \ (v \otimes \lambda) \circ (w \otimes \mu) = \text{tr} (\lambda(w) v \otimes \mu) = \lambda(w) \text{tr}(v \otimes \mu) = \lambda(w) \mu(v) \]

The right-hand side is

\[ \text{tr} \ (w \otimes \mu) \circ (v \otimes \lambda) = \text{tr} (\mu(v) w \otimes \lambda) = \mu(v) \text{tr}(w \otimes \lambda) = \mu(v) \lambda(w) \]

These elements of \( k \) are the same. //

\[08.40\] Prove the \textit{expansion by minors} formula for determinants, namely, for an \( n \)-by-\( n \) matrix \( A \) with entries \( a_{ij} \), letting \( A^{ij} \) be the matrix obtained by deleting the \( i^{th} \) row and \( j^{th} \) column, for any fixed row index \( i \),

\[ \det A = (-1)^i \sum_{j=1}^{n} (-1)^j a_{ij} \det A^{ij} \]

and symmetrically for expansion along a column.

**Discussion:** [iou: prove that this formula is linear in each row/column, and invoke the uniqueness of determinants]

\[08.41\] Let \( M \) and \( N \) be free \( R \)-modules, where \( R \) is a commutative ring with identity. Prove that \( M \otimes_R N \) is free and

\[ \text{rank} M \otimes_R N = \text{rank} M \cdot \text{rank} N \]

**Discussion:** Let \( M \) and \( N \) be free on generators \( i : X \rightarrow M \) and \( j : Y \rightarrow N \). We claim that \( M \otimes_R N \) is free on a set map

\[ \ell : X \times Y \rightarrow M \otimes_R N \]
To verify this, let $\varphi : X \times Y \to Z$ be a set map. For each fixed $y \in Y$, the map $x \to \varphi(x, y)$ factors through a unique $R$-module map $B_y : M \to Z$. For each $m \in M$, the map $y \to B_y(m)$ gives rise to a unique $R$-linear map $n \to B(m, n)$ such that

$$B(m, j(y)) = B_y(m)$$

The linearity in the second argument assures that we still have the linearity in the first, since for $n = \sum t_r j(y_t)$ we have

$$B(m, n) = B(m, \sum t_r j(y_t)) = \sum t_r B_y(m)$$

which is a linear combination of linear functions. Thus, there is a unique map to $Z$ induced on the tensor product, showing that the tensor product with set map $i \times j : X \times Y \to M \otimes_R N$ is free. ///

[08.42] Let $M$ be a free $R$-module of rank $r$, where $R$ is a commutative ring with identity. Let $S$ be a commutative ring with identity containing $R$, such that $1_R = 1_S$. Prove that as an $S$ module $M \otimes_R S$ is free of rank $r$.

Discussion: We prove a bit more. First, instead of simply an inclusion $R \subset S$, we can consider any ring homomorphism $\psi : R \to S$ such that $\psi(1_R) = 1_S$.

Also, we can consider arbitrary sets of generators, and give more details. Let $M$ be free on generators $i : X \to M$, where $X$ is a set. Let $\tau : M \times S \to M \otimes_R S$ be the canonical map. We claim that $M \otimes_R S$ is free on $j : X \to M \otimes_R S$ defined by

$$j(x) = \tau(i(x) \times 1_S)$$

Given an $S$-module $N$, we can be a little forgetful and consider $N$ as an $R$-module via $\psi$, by $r \cdot n = \psi(r)n$. Then, given a set map $\varphi : X \to N$, since $M$ is free, there is a unique $R$-module map $\Phi : M \to N$ such that $\varphi = \Phi \circ i$. That is, the diagram

```
    X
   ↑ i
  M  φ
    ↓
   N
```

commutes. Then the map

$$\psi : M \times S \to N$$

by

$$\psi(m \times s) = s \cdot \Phi(m)$$

induces (by the defining property of $M \otimes_R S$) a unique $\Psi : M \otimes_R S \to N$ making a commutative diagram

```
    M \otimes_R S
   ↓ \tau
  M \times S
  ↓ i \times inc
X \times \{1_S\}  \psi
    ↓
   X  φ
    ↓
  N
```

where $inc$ is the inclusion map $\{1_S\} \to S$, and where $t : X \to X \times \{1_S\}$ by $x \to x \times 1_S$. Thus, $M \otimes_R S$ is free on the composite $j : X \to M \otimes_R S$ defined to be the composite of the vertical maps in that last diagram. This argument does not depend upon finiteness of the generating set. ///
For finite-dimensional vector spaces $V, W$ over a field $k$, prove that there is a natural isomorphism

$$(V \otimes_k W)^* \approx V^* \otimes W^*$$

where $X^* = \text{Hom}_k(X, k)$ for a $k$-vectorspace $X$.

**Discussion:** For finite-dimensional $V$ and $W$, since $V \otimes_k W$ is free on the cartesian product of the generators for $V$ and $W$, the dimensions of the two sides match. We make an isomorphism from right to left. Create a bilinear map

$$V^* \times W^* \to (V \otimes_k W)^*$$

as follows. Given $\lambda \in V^*$ and $\mu \in W^*$, as usual make $\Lambda_{\lambda, \mu} \in (V \otimes_k W)^*$ from the bilinear map

$$B_{\lambda, \mu} : V \times W \to k$$

defined by

$$B_{\lambda, \mu}(v, w) = \lambda(v) \cdot \mu(w)$$

This induces a unique functional $\Lambda_{\lambda, \mu}$ on the tensor product. This induces a unique linear map

$$V^* \otimes W^* \to (V \otimes_k W)^*$$

as desired.

Since everything is finite-dimensional, bijectivity will follow from injectivity. Let $e_1, \ldots, e_m$ be a basis for $V$, $f_1, \ldots, f_n$ a basis for $W$, and $\lambda_1, \ldots, \lambda_m$ and $\mu_1, \ldots, \mu_n$ corresponding dual bases. We have shown that a basis of a tensor product of free modules is free on the cartesian product of the generators. Suppose that $\sum_{ij} c_{ij} \lambda_i \otimes \mu_j$ gives the 0 functional on $V \otimes W$, for some scalars $c_{ij}$. Then, for every pair of indices $s, t$, the function is 0 on $e_s \otimes f_t$. That is,

$$0 = \sum_{ij} c_{ij} \lambda_i(e_s) \cdot \lambda_j(f_t) = c_{st}$$

Thus, all constants $c_{ij}$ are 0, proving that the map is injective. Then a dimension count proves the isomorphism.

For a finite-dimensional $k$-vectorspace $V$, prove that the bilinear map

$$B : V \times V^* \to \text{End}_k(V)$$

by

$$B(v \times \lambda)(x) = \lambda(x) \cdot v$$

gives an isomorphism $V \otimes_k V^* \to \text{End}_k(V)$. Further, show that the composition of endomorphisms is the same as the map induced from the map on

$$(V \otimes V^*) \times (V \otimes V^*) \to V \otimes V^*$$

given by

$$(v \otimes \lambda) \times (w \otimes \mu) \to \lambda(w) v \otimes \mu$$

**Discussion:** The bilinear map $v \times \lambda \to T_{v, \lambda}$ given by

$$T_{v, \lambda}(w) = \lambda(w) \cdot v$$

induces a unique linear map $j : V \otimes V^* \to \text{End}_k(V)$. 

To prove that $j$ is injective, we may use the fact that a basis of a tensor product of free modules is free on the cartesian product of the generators. Thus, let $e_1, \ldots, e_n$ be a basis for $V$, and $\lambda_1, \ldots, \lambda_n$ a dual basis for $V^*$. Suppose that

$$\sum_{i,j=1}^n c_{ij} e_i \otimes \lambda_j \rightarrow 0 \text{End}_k(V)$$

That is, for every $e_\ell$,

$$\sum_{ij} c_{ij} \lambda_j(e_\ell)e_i = 0 \in V$$

This is

$$\sum_{i} c_{ij}e_i = 0 \quad (\text{for all } j)$$

Since the $e_i$s are linearly independent, all the $c_{ij}$s are 0. Thus, the map $j$ is injective. Then counting $k$-dimensions shows that this $j$ is a $k$-linear isomorphism.

Composition of endomorphisms is a bilinear map

$$\text{End}_k(V) \times \text{End}_k(V) \rightarrow \text{End}_k(V)$$

by

$$S \times T \rightarrow S \circ T$$

Denote by

$$c : (v \otimes \lambda) \times (w \otimes \mu) \rightarrow \lambda(w)v \otimes \mu$$

the allegedly corresonding map on the tensor products. The induced map on $(V \otimes V^*) \otimes (V \otimes V^*)$ is an example of a contraction map on tensors. We want to show that the diagram

$$\begin{array}{ccc}
\text{End}_k(V) \times \text{End}_k(V) & \xrightarrow{\circ} & \text{End}_k(V) \\
\downarrow j \times j & & \downarrow j \\
(V \otimes_k V^*) \times (V \otimes_k V^*) & \xrightarrow{c} & V \otimes_k V^*
\end{array}$$

commutes. It suffices to check this starting with $(v \otimes \lambda) \times (w \otimes \mu)$ in the lower left corner. Let $x \in V$. Going up, then to the right, we obtain the endomorphism which maps $x$ to

$$j(v \otimes \lambda) \circ j(w \otimes \mu)(x) = j(v \otimes \lambda)(j(w \otimes \mu)(x)) = j(v \otimes \lambda)(\mu(x)w) = \mu(x)\lambda(w)v$$

Going the other way around, to the right then up, we obtain the endomorphism which maps $x$ to

$$j(c((v \otimes \lambda) \times (w \otimes \mu)))(x) = j(\lambda(w)(v \otimes \mu))(x) = \lambda(w)\mu(x)v$$

These two outcomes are the same. 

[08.45] Via the isomorphism $\text{End}_k(V) \cong V \otimes_k V^*$, show that the linear map

$$\text{tr} : \text{End}_k(V) \rightarrow k$$

is the linear map

$$V \otimes V^* \rightarrow k$$

induced by the bilinear map $v \times \lambda \rightarrow \lambda(v)$. 

24
Discussion: Note that the induced map

\[ V \otimes_k V^* \to k \text{ by } v \otimes \lambda \to \lambda(v) \]

is another contraction map on tensors. Part of the issue is to compare the coordinate-bound trace with the induced (contraction) map \( t(v \otimes \lambda) = \lambda(v) \) determined uniquely from the bilinear map \( v \times \lambda \to \lambda(v) \). To this end, let \( e_1, \ldots, e_n \) be a basis for \( V \), with dual basis \( \lambda_1, \ldots, \lambda_n \). The corresponding matrix coefficients \( T_{ij} \in k \) of a \( k \)-linear endomorphism \( T \) of \( V \) are

\[ T_{ij} = \lambda_i(Te_j) \]

(Always there is the worry about interchange of the indices.) Thus, in these coordinates,

\[ \text{tr}T = \sum_i \lambda_i(Te_i) \]

Let \( T = j(e_s \otimes \lambda_t) \). Then, since \( \lambda_t(e_i) = 0 \) unless \( i = t \),

\[ \text{tr}T = \sum_i \lambda_i(Te_i) = \sum_i \lambda_i(j(e_s \otimes \lambda_t)e_i) = \sum_i \lambda_i(\lambda_t(e_i) \cdot e_s) = \lambda_t(\lambda_t(\lambda_t(e_i) \cdot e_s)) = \begin{cases} 1 & (s = t) \\ 0 & (s \neq t) \end{cases} \]

On the other hand,

\[ t(e_s \otimes \lambda_t) = \lambda_t(e_s) = \begin{cases} 1 & (s = t) \\ 0 & (s \neq t) \end{cases} \]

Thus, these two \( k \)-linear functionals agree on the monomials, which span, they are equal. ///

[08.46] Prove that \( \text{tr}(AB) = \text{tr}(BA) \) for two endomorphisms of a finite-dimensional vector space \( V \) over a field \( k \), with trace defined as just above.

Discussion: Since the maps \( \text{End}_k(V) \times \text{End}_k(V) \to k \) by

\[ A \times B \to \text{tr}(AB) \quad \text{and/or} \quad A \times B \to \text{tr}(BA) \]

are bilinear, it suffices to prove the equality on (images of) monomials \( v \otimes \lambda \), since these span the endomorphisms over \( k \). Previous examples have converted the issue to one concerning \( V \otimes_k V^* \). (We have already shown that the isomorphism \( V \otimes_k V^* \approx \text{End}_k(V) \) is converts a contraction map on tensors to composition of endomorphisms, and that the trace on tensors defined as another contraction corresponds to the trace of matrices.) Let \( \text{tr} \) now denote the contraction-map trace on tensors, and (temporarily) write

\( (v \otimes \lambda) \circ (w \otimes \mu) = \lambda(w) v \otimes \mu \)

for the contraction-map composition of endomorphisms. Thus, we must show that

\[ \text{tr} (v \otimes \lambda) \circ (w \otimes \mu) = \text{tr} (w \otimes \mu) \circ (v \otimes \lambda) \]

The left-hand side is

\[ \text{tr} (v \otimes \lambda) \circ (w \otimes \mu) = \text{tr}(\lambda(w) v \otimes \mu) = \lambda(w) \text{tr}(v \otimes \mu) = \lambda(w) \mu(v) \]

The right-hand side is

\[ \text{tr} (w \otimes \mu) \circ (v \otimes \lambda) = \text{tr}(\mu(v) w \otimes \lambda) = \mu(v) \text{tr}(w \otimes \lambda) = \mu(v) \lambda(w) \]

These elements of \( k \) are the same. ///
Prove that tensor products are associative, in the sense that, for $R$-modules $A, B, C$, we have a natural isomorphism

$$A \otimes_R (B \otimes_R C) \cong (A \otimes_R B) \otimes_R C$$

In particular, do prove the naturality, at least the one-third part of it which asserts that, for every $R$-module homomorphism $f : A \to A'$, the diagram

$$
\begin{array}{ccc}
A \otimes_R (B \otimes_R C) & \cong & (A \otimes_R B) \otimes_R C \\
\downarrow f \otimes (1_B \otimes 1_C) & & \downarrow (f \otimes 1_B) \otimes 1_C \\
A' \otimes_R (B \otimes_R C) & \cong & (A' \otimes_R B) \otimes_R C
\end{array}
$$

commutes, where the two horizontal isomorphisms are those determined in the first part of the problem. (One might also consider maps $g : B \to B'$ and $h : C \to C'$, but these behave similarly, so there’s no real compulsion to worry about them, apart from awareness of the issue.)

**Discussion:** Since all tensor products are over $R$, we drop the subscript, to lighten the notation. As usual, to make a (linear) map from a tensor product $M \otimes N$, we induce uniquely from a bilinear map on $M \times N$. We have done this enough times that we will suppress this part now.

The thing that is slightly less trivial is construction of maps to tensor products $M \otimes N$. These are always obtained by composition with the canonical bilinear map

$$M \times N \to M \otimes N$$

Important at present is that we can create $n$-fold tensor products, as well. Thus, we prove the indicated isomorphism by proving that both the indicated iterated tensor products are (naturally) isomorphic to the un-parenthesis’d tensor product $A \otimes B \otimes C$, with canonical map $\tau : A \times B \times C \to A \otimes B \otimes C$, such that for every trilinear map $\varphi : A \times B \times C \to X$ there is a unique linear $\Phi : A \otimes B \otimes C \to X$ such that

$$
\begin{array}{ccc}
A \otimes B \otimes C & \xrightarrow{\tau} & (A \otimes B) \otimes C \\
\downarrow & & \downarrow \\
A \times B \times C & \xrightarrow{\varphi} & X
\end{array}
$$

The set map

$$A \times B \times C \approx (A \times B) \times C \to (A \otimes B) \otimes C$$

by

$$a \times b \times c \to (a \times b) \times c \to (a \otimes b) \otimes c$$

is linear in each single argument (for fixed values of the others). Thus, we are assured that there is a unique induced linear map

$$A \otimes B \otimes C \to (A \otimes B) \otimes C$$

such that

$$
\begin{array}{ccc}
A \otimes B \otimes C & \xrightarrow{\tau} & (A \otimes B) \otimes C \\
\downarrow & & \\
A \times B \times C & \xrightarrow{\varphi} & (A \otimes B) \otimes C
\end{array}
$$

commutes.

Similarly, from the set map

$$(A \times B) \times C \approx A \times B \times C \to A \otimes B \otimes C$$
by
\[(a \times b) \times c \rightarrow a \times b \times c \rightarrow a \otimes b \otimes c\]
is linear in each single argument (for fixed values of the others). Thus, we are assured that there is a unique induced linear map
\[(A \otimes B) \otimes C \rightarrow A \otimes B \otimes C\]
such that
\[(A \otimes B) \otimes C \rightarrow A \otimes B \otimes C\]
commutes.

Then \(j \circ i\) is a map of \(A \otimes B \otimes C\) to itself compatible with the canonical map \(A \times B \times C \rightarrow A \otimes B \otimes C\). By uniqueness, \(j \circ i\) is the identity on \(A \otimes B \otimes C\). Similarly (just very slightly more complicatedly), \(i \circ j\) must be the identity on the iterated tensor product. Thus, these two maps are mutual inverses.

To prove naturality in one of the arguments \(A,B,C\), consider \(f : C \rightarrow C'\). Let \(j_{ABC}\) be the isomorphism for a fixed triple \(A,B,C\), as above. The diagram of maps of cartesian products (of sets, at least)
\[
\begin{array}{ccc}
(A \times B) \times C & \xrightarrow{j_{ABC}} & A \times B \times C \\
(1_A \times 1_B \times f) \downarrow & & \downarrow 1_A \times 1_B \times f \\
(A \times B) \times C & \xrightarrow{j} & A \times B \times C
\end{array}
\]
does commute: going down, then right, is
\[
j_{ABC'}((1_A \times 1_B) \times f)((a \times b) \times c)) = j_{ABC'}((a \times b) \times f(c)) = a \times b \times f(c)
\]
Going right, then down, gives
\[
(1_A \times 1_B \times f)(j_{ABC}((a \times b) \times c)) = (1_A \times 1_B \times f)(a \times b \times c)) = a \times b \times f(c)
\]
These are the same.

[08.48] Consider the injection \(\mathbb{Z}/2 \rightarrow \mathbb{Z}/4\) which maps
\[
t : x + 2\mathbb{Z} \rightarrow 2x + 4\mathbb{Z}
\]
Show that the induced map
\[
t \otimes 1_{\mathbb{Z}/2} : \mathbb{Z}/2 \otimes_{\mathbb{Z}} \mathbb{Z}/2 \rightarrow \mathbb{Z}/4 \otimes_{\mathbb{Z}} \mathbb{Z}/2
\]
is no longer an injection.

**Discussion:** We claim that \(t \otimes 1\) is the 0 map. Indeed,
\[
(t \otimes 1)(m \otimes n) = 2m \otimes n = 2 \cdot (m \otimes n) = m \otimes 2n = m \otimes 0 = 0
\]
for all \(m \in \mathbb{Z}/2\) and \(n \in \mathbb{Z}/2\).

[08.49] Prove that if \(s : M \rightarrow N\) is a surjection of \(\mathbb{Z}\)-modules and \(X\) is any other \(\mathbb{Z}\) module, then the induced map
\[
s \otimes 1_X : M \otimes_{\mathbb{Z}} X \rightarrow N \otimes_{\mathbb{Z}} X
\]
is still surjective.

Discussion: Given \( \sum_i n_i \otimes x_i \) in \( N \otimes \mathbb{Z} X \), let \( m_i \in M \) be such that \( s(m_i) = n_i \). Then

\[
(s \otimes 1)(\sum_i m_i \otimes x_i) = \sum_i s(m_i) \otimes x_i = \sum_i n_i \otimes x_i
\]

so the map is surjective. 

[0.6] Remark: Note that the only issue here is hidden in the verification that the induced map \( s \otimes 1 \) exists.

[08.50] Give an example of a surjection \( f : M \to N \) of \( \mathbb{Z} \)-modules, and another \( \mathbb{Z} \)-module \( X \), such that the induced map

\[
f \circ - : \text{Hom}_\mathbb{Z}(X, M) \to \text{Hom}_\mathbb{Z}(X, N)
\]

(by post-composing) fails to be surjective.

Discussion: Let \( M = \mathbb{Z} \) and \( N = \mathbb{Z}/n \) with \( n > 0 \). Let \( X = \mathbb{Z}/n \). Then

\[
\text{Hom}_\mathbb{Z}(X, M) = \text{Hom}_\mathbb{Z}(\mathbb{Z}/n, \mathbb{Z}) = 0
\]

since

\[
0 = \varphi(0) = \varphi(nx) = n \cdot \varphi(x) \in \mathbb{Z}
\]

so (since \( n \) is not a 0-divisor in \( \mathbb{Z} \)) \( \varphi(x) = 0 \) for all \( x \in \mathbb{Z}/n \). On the other hand,

\[
\text{Hom}_\mathbb{Z}(X, N) = \text{Hom}_\mathbb{Z}(\mathbb{Z}/n, \mathbb{Z}/n) \approx \mathbb{Z}/n \neq 0
\]

Thus, the map cannot possibly be surjective.

[08.51] Let \( G : \{ \mathbb{Z} \text{-modules} \} \to \{ \text{sets} \} \) be the functor that forgets that a module is a module, and just retains the underlying set. Let \( F : \{ \text{sets} \} \to \{ \mathbb{Z} \text{-modules} \} \) be the functor which creates the free module \( FS \) on the set \( S \) (and keeps in mind a map \( i : S \to FS \)). Show that for any set \( S \) and any \( \mathbb{Z} \)-module \( M \)

\[
\text{Hom}_\mathbb{Z}(FS, M) \approx \text{Hom}_{\text{sets}}(S, GM)
\]

Prove that the isomorphism you describe is natural in \( S \). (It is also natural in \( M \), but don’t prove this.)

Discussion: Our definition of free module says that \( FS = X \) is free on a (set) map \( i : S \to X \) if for every set map \( \varphi : S \to M \) with \( R \)-module \( M \) gives a unique \( R \)-module map \( \Phi : X \to M \) such that the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\Phi} & M \\
\downarrow{\Phi} & \nearrow{\varphi} \\
S & \nearrow{\varphi} & M
\end{array}
\]

commutes. Of course, given \( \Phi \), we obtain \( \varphi = \Phi \circ i \) by composition (in effect, restriction). We claim that the required isomorphism is

\[
\text{Hom}_\mathbb{Z}(FS, M) \leftrightarrow \text{Hom}_{\text{sets}}(S, GM)
\]

Even prior to naturality, we must prove that this is a bijection. Note that the set of maps of a set into an \( R \)-module has a natural structure of \( R \)-module, by

\[
(r \cdot \varphi)(s) = r \cdot \varphi(s)
\]
The map in the direction $\varphi \to \Phi$ is an injection, because two maps $\varphi, \psi$ mapping $S \to M$ that induce the same map $\Phi$ on $X$ give $\varphi = \Phi \circ i = \psi$, so $\varphi = \psi$. And the map $\varphi \to \Phi$ is surjective because a given $\Phi$ is induced from $\varphi = \Phi \circ i$.

For naturality, for fixed $S$ and $M$ let the map $\varphi \to \Phi$ be named $j_{S,M}$. That is, the isomorphism is

$$\operatorname{Hom}_\mathbb{Z}(FS, M) \xleftarrow{j_{S,X}} \operatorname{Hom}_{\text{sets}}(S, GM)$$

To show naturality in $S$, let $f : S \to S'$ be a set map. Let $i' : S' \to X'$ be a free module on $S'$. That is, $X' = FS'$. We must show that

$$\operatorname{Hom}_\mathbb{Z}(FS, M) \xleftarrow{j_{S,M}} \operatorname{Hom}_{\text{sets}}(S, GM)$$

commutes, where $- \circ f$ is pre-composition by $f$, and $- \circ Ff$ is pre-composition by the induced map $Ff : FS \to FS'$ on the free modules $X = FS$ and $X' = FS'$. Let $\varphi \in \operatorname{Hom}_{\text{sets}}(S', GM)$, and $x = \sum_s r_s \cdot i(s) \in X = FS$. Go up, then left, in the diagram, computing,

$$(j_{S,M} \circ (- \circ f))(\varphi)(x) = j_{S,M}(\varphi \circ f)(x) = j_{S,M}(\varphi \circ f)\left(\sum_s r_s \cdot i(s)\right) = \sum_s r_s (\varphi \circ f)(s)$$

On the other hand, going left, then up, gives

$$((- \circ Ff) \circ j_{S', M})(\varphi)(x) = (j_{S',M}(\varphi \circ Ff)(x) = (j_{S',M}(\varphi))(Ff(x)$$

$$= (j_{S',M}(\varphi))\left(\sum_s r_s' \cdot i'(fs)\right) = \sum_s r_s \varphi(fs)$$

These are the same.

**Discussion:** This is the easiest of this and the following two examples. Namely, let $M_i$ be the 2-by-2 matrix obtained by omitting the $i^{th}$ column of the given matrix. Let $a, b, c$ be integers such that

$$a \det M_1 - b \det M_2 + c \det M_3 = \gcd(\det M_1, \det M_2, \det M_3) = 1$$

Then, expanding by minors,

$$\det \begin{pmatrix} a & b & c \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix} = a \det M_1 - b \det M_2 + c \det M_3 = 1$$

as desired.
Let $a, b, c$ be integers whose $\gcd$ is 1. Prove (without manipulating matrices) that there is a 3-by-3 integer matrix with top row $(a \ b \ c)$ with determinant 1.

**Discussion:** Let $F = \mathbb{Z}^3$, and $E = \mathbb{Z} \cdot (a, b, c)$. We claim that, since $\gcd(a, b, c) = 1$, $F/E$ is torsion-free. Indeed, for $(x, y, z) \in F = \mathbb{Z}^3$, $r \in \mathbb{Z}$, and $r \cdot (x, y, z) \in E$, there must be an integer $t$ such that $ta = rx$, $tb = ry$, and $tc = rz$. Let $u, v, w$ be integers such that
\[
ua + vb + wz = \gcd(a, b, c) = 1
\]
Then the usual stunt gives
\[
t = t \cdot 1 = t \cdot (ua + vb + wz) = u(ta) + v(tb) + w(tc) = u(rx) + v(ry) + w(rz) = r \cdot (ux + vy + wz)
\]
This implies that $r | t$. Thus, dividing through by $r$, $(x, y, z) \in \mathbb{Z} \cdot (a, b, c)$, as claimed.

Invoking the Structure Theorem for finitely-generated $\mathbb{Z}$-modules, there is a basis $f_1, f_2, f_3$ for $F$ and $0 < d_1 \in \mathbb{Z}$ such that $E = \mathbb{Z} \cdot d_1 f_1$. Since $F/E$ is torsionless, $d_1 = 1$, and $E = \mathbb{Z} \cdot f_1$. Further, since both $(a, b, c)$ and $f_1$ generate $E$, and $\mathbb{Z}^* = \{ \pm 1 \}$, without loss of generality we can suppose that $f_1 = (a, b, c)$.

Let $A$ be an endomorphism of $F = \mathbb{Z}^3$ such that $Af_i = e_i$. Then, writing $A$ for the matrix giving the endomorphism $A$,
\[
(a, b, c) \cdot A = (1, 0, 0)
\]
Since $A$ has an inverse $B$,
\[
1 = \det 1_3 = \det (AB) = \det A \cdot \det B
\]
so the determinants of $A$ and $B$ are in $\mathbb{Z}^* = \{ \pm 1 \}$. We can adjust $A$ by right-multiplying by
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{pmatrix}
\]
to make $\det A = +1$, and retaining the property $f_1 \cdot A = e_1$. Then
\[
A^{-1} = 1_3 \cdot A^{-1} = \begin{pmatrix} e_1 & e_2 & e_3 \end{pmatrix} \cdot A^{-1} = \begin{pmatrix} a & b & c \\
* & * & * \\
* & * & *
\end{pmatrix}
\]
That is, the original $(a, b, c)$ is the top row of $A^{-1}$, which has integer entries and determinant 1. ///

Let $M = \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} & m_{15} \\
m_{21} & m_{22} & m_{23} & m_{24} & m_{25} \\
m_{31} & m_{32} & m_{33} & m_{34} & m_{35} \end{pmatrix}$ and suppose that the $\gcd$ of all determinants of 3-by-3 minors is 1. Prove that there exists a 5-by-5 integer matrix $\tilde{M}$ with $M$ as its top 3 rows, such that $\det \tilde{M} = 1$.

**Discussion:** Let $F = \mathbb{Z}^5$, and let $E$ be the submodule generated by the rows of the matrix. Since $\mathbb{Z}$ is a PID and $F$ is free, $E$ is free.

Let $e_1, \ldots, e_5$ be the standard basis for $\mathbb{Z}^5$. We have shown that the monomials $e_{i_1} \wedge e_{i_2} \wedge e_{i_3}$ with $i_1 < i_2 < i_3$ are a basis for $\wedge^3 F$. Since the $\gcd$ of the determinants of 3-by-3 minors is 1, some determinant of 3-by-3 minor is non-zero, so the rows of $M$ are linearly independent over $\mathbb{Q}$, so $E$ has rank 3 (rather than something less). The structure theorem tells us that there is a $\mathbb{Z}$-basis $f_1, \ldots, f_5$ for $F$ and divisors $d_1 | d_2 | d_3$ (all non-zero since $E$ is of rank 3) such that
\[
E = \mathbb{Z} \cdot d_1 f_1 \oplus \mathbb{Z} \cdot d_2 f_2 \oplus \mathbb{Z} \cdot d_3 f_3
\]
Let \( i : E \to F \) be the inclusion. Consider \( \land^3 : \land^3 E \to \land^3 F \). We know that \( \land^3 E \) has \( \mathbb{Z} \)-basis

\[
d_1 f_1 \land d_2 f_2 \land d_3 f_3 = (d_1 d_2 d_3) \cdot (f_1 \land f_2 \land f_3)
\]

On the other hand, we claim that the coefficients of \( (d_1 d_2 d_3) \cdot (f_1 \land f_2 \land f_3) \) in terms of the basis \( e_{i_1} \land e_{i_2} \land e_{i_3} \) for \( \land^3 F \) are exactly (perhaps with a change of sign) the determinants of the 3-by-3 minors of \( M \). Indeed, since both \( f_1, f_2, f_3 \) and the three rows of \( M \) are bases for the rowspace of \( M \), the \( f_i \)'s are linear combinations of the rows, and vice-versa (with integer coefficients). Thus, there is a 3-by-3 matrix with determinant \( \pm 1 \) such that left multiplication of \( M \) by it yields a new matrix with rows \( f_1, f_2, f_3 \). At the same time, this changes the determinants of 3-by-3 minors by at most \( \pm 1 \), by the multiplicativity of determinants.

The hypothesis that the \( \gcd \) of all these coordinates is 1 means exactly that \( \land^3 F / \land^3 E \) is torsion-free. (If the coordinates had a common factor \( d > 1 \), then \( d \) would annihilate the quotient.) This requires that \( d_1 d_2 d_3 = 1 \), so \( d_1 = d_2 = d_3 = 1 \) (since we take these divisors to be positive). That is, \( E = \mathbb{Z} \cdot f_1 \oplus \mathbb{Z} \cdot f_2 \oplus \mathbb{Z} \cdot f_3 \)

Writing \( f_1, f_2, \) and \( f_3 \) as row vectors, they are \( \mathbb{Z} \)-linear combinations of the rows of \( M \), which is to say that there is a 3-by-3 integer matrix \( L \) such that

\[
L \cdot M = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}
\]

Since the \( f_i \) are also a \( \mathbb{Z} \)-basis for \( E \), there is another 3-by-3 integer matrix \( K \) such that

\[
M = K \cdot \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}
\]

Then \( LK = KL = 1_3 \). In particular, taking determinants, both \( K \) and \( L \) have determinants in \( \mathbb{Z}^\times \), namely, \( \pm 1 \).

Let \( A \) be a \( \mathbb{Z} \)-linear endomorphism of \( F = \mathbb{Z}^5 \) mapping \( f_i \) to \( e_i \). Also let \( A \) be the 5-by-5 integer matrix such that right multiplication of a row vector by \( A \) gives the effect of the endomorphism \( A \). Then

\[
L \cdot M \cdot A = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} \cdot A = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}
\]

Since the endomorphism \( A \) is invertible on \( F = \mathbb{Z}^5 \), it has an inverse endomorphism \( A^{-1} \), whose matrix has integer entries. Then

\[
M = L^{-1} \cdot \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} \cdot A^{-1}
\]

Let

\[
\Lambda = \begin{pmatrix} L^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \pm 1 \end{pmatrix}
\]

where the \( \pm 1 = \det A = \det A^{-1} \). Then

\[
\Lambda \cdot \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{pmatrix} \cdot A^{-1} = \Lambda \cdot 1_5 \cdot A^{-1} = \Lambda \cdot A^{-1}
\]

31
has integer entries and determinant 1 (since we adjusted the ±1 in Λ). At the same time, it is
\[
Λ \cdot A^{-1} = \begin{pmatrix} L^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \pm 1 \end{pmatrix} \cdot \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} \cdot A^{-1} = \begin{pmatrix} M \\ * \\ * \end{pmatrix} = 5\text{-by-}5
\]
This is the desired integer matrix \( \tilde{M} \) with determinant 1 and upper 3 rows equal to the given matrix.

[08.55] Let \( R \) be a commutative ring with unit. For a finitely-generated free \( R \)-module \( F \), prove that there is a (natural) isomorphism
\[
\text{Hom}_R(F, R) \cong F
\]
Or is it only
\[
\text{Hom}_R(R, F) \cong F
\]
instead? (Hint: Recall the definition of a free module.)

Discussion: For any \( R \)-module \( M \), there is a (natural) isomorphism
\[
i : M \rightarrow \text{Hom}_R(R, M)
\]
given by
\[
i(m)(r) = r \cdot m
\]
This is injective, since if \( i(m)(r) \) were the 0 homomorphism, then \( i(m)(r) = 0 \) for all \( r \), which is to say that \( r \cdot m = 0 \) for all \( r \in R \), in particular, for \( r = 1 \). Thus, \( m = 1 \cdot m = 0 \), so \( m = 0 \). (Here we use the standing assumption that \( 1 \cdot m = m \) for all \( m \in M \).) The map is surjective, since, given \( \varphi \in \text{Hom}_R(R, M) \), we have
\[
\varphi(r) = \varphi(r \cdot 1) = r \cdot \varphi(1)
\]
That is, \( m = \varphi(1) \) determines \( \varphi \) completely. Then \( \varphi = i(\varphi(m)) \) and \( m = i(m)(1) \), so these are mutually inverse maps. This did not use finite generation, nor free-ness.

Consider now the other form of the question, namely whether or not
\[
\text{Hom}_R(F, R) \cong F
\]
is valid for \( F \) finitely-generated and free. Let \( F \) be free on \( i : S \rightarrow F \), with finite \( S \). Use the natural isomorphism
\[
\text{Hom}_R(F, R) \cong \text{Hom}_{\text{sets}}(S, R)
\]
discussed earlier. The right-hand side is the collection of \( R \)-valued functions on \( S \). Since \( S \) is finite, the collection of all \( R \)-valued functions on \( S \) is just the collection of functions which vanish off a finite subset. The latter was our construction of the free \( R \)-module on \( S \). So we have the isomorphism.

[0.7] Remark: Note that if \( S \) is not finite, \( \text{Hom}_R(F, R) \) is too large to be isomorphic to \( F \). If \( F \) is not free, it may be too small. Consider \( F = \mathbb{Z}/n \) and \( R = \mathbb{Z} \), for example.

[0.8] Remark: And this discussion needs a choice of the generators \( i : S \rightarrow F \). In the language style which speaks of generators as being chosen elements of the module, we have most certainly chosen a basis.

[08.56] Let \( R \) be an integral domain. Let \( M \) and \( N \) be free \( R \)-modules of finite ranks \( r, s \), respectively. Suppose that there is an \( R \)-bilinear map
\[
B : M \times N \rightarrow R
\]
which is non-degenerate in the sense that for every \(0 \neq m \in M\) there is \(n \in N\) such that \(B(m, n) \neq 0\), and vice-versa. Prove that \(r = s\).

**Discussion:** All tensors and homomorphisms are over \(R\), so we suppress the subscript and other references to \(R\) when reasonable to do so. We use the important natural isomorphism (proven afterward)

\[
\text{Hom}(A \otimes B, C) \xrightarrow{i_{A,B,C}} \text{Hom}(A, \text{Hom}(B, C))
\]

by

\[
i_{A,B,C}(\Phi)(a)(b) = \Phi(a \otimes b)
\]

We also use the fact (from an example just above) that for \(F\) free on \(t : S \to F\) there is the natural (given \(t : S \to F\), anyway!) isomorphism

\[
j : \text{Hom}(F, R) \approx \text{Hom}_{\text{sets}}(S, R) = F
\]

for modules \(E\), given by

\[
j(\psi)(s) = \psi(t(s))
\]

where we use construction of free modules on sets \(S\) that they are \(R\)-valued functions on \(S\) taking non-zero values at only finitely-many elements.

Thus,

\[
\text{Hom}(M \otimes N, R) \xrightarrow{i} \text{Hom}(M, \text{Hom}(N, R)) \xrightarrow{j} \text{Hom}(M, N)
\]

The bilinear form \(B\) induces a linear functional \(\beta\) such that

\[
\beta(m \otimes n) = B(m, n)
\]

The hypothesis says that for each \(m \in M\) there is \(n \in N\) such that

\[
i(\beta)(m)(n) \neq 0
\]

That is, for all \(m \in M\), \(i(\beta)(m) \in \text{Hom}(N, R) \approx N = 0\). That is, the map \(m \to i(\beta)(m)\) is injective. So the existence of the non-degenerate bilinear pairing yields an injection of \(M\) to \(N\). Symmetrically, there is an injection of \(N\) to \(M\).

Using the assumption that \(R\) is a PID, we know that a submodule of a free module is free of lesser-or-equal rank. Thus, the two inequalities

\[
\text{rank} M \leq \text{rank} N \quad \text{rank} N \leq \text{rank} M
\]

from the two inclusions imply equality.

///

**[0.9] Remark:** The hypothesis that \(R\) is a PID may be too strong, but I don’t immediately see a way to work around it.

Now let’s prove (again?) that

\[
\text{Hom}(A \otimes B, C) \xrightarrow{i} \text{Hom}(A, \text{Hom}(B, C))
\]

by

\[
i(\Phi)(a)(b) = \Phi(a \otimes b)
\]

is an isomorphism. The map in the other direction is

\[
j(\varphi)(a \otimes b) = \varphi(a)(b)
\]
First,
\[ i(j(\varphi))(a)(b) = j(\varphi)(a \otimes b) = \varphi(a)(b) \]

Second,
\[ j(i(\Phi))(a \otimes b) = i(\Phi)(a)(b) = \Phi(a \otimes b) \]

Thus, these maps are mutual inverses, so each is an isomorphism.

Let \( \varphi : R \to S \) be commutative rings with unit, and suppose that \( \varphi(1_R) = 1_S \), thus making \( S \) an \( R \)-algebra. For an \( R \)-module \( N \) prove that \( \text{Hom}_R(S,N) \) is (yet another) good definition of extension of scalars from \( R \) to \( S \), by checking that for every \( S \)-module \( M \) there is a natural isomorphism
\[
\text{Hom}_R(\text{Res}_R^S M, N) \approx \text{Hom}_S(M, \text{Hom}_R(S,N))
\]
where \( \text{Res}_R^S M \) is the \( R \)-module obtained by forgetting \( S \), and letting \( r \in R \) act on \( M \) by \( r \cdot m = \varphi(r)m \). (Do prove naturality in \( M \), also.)

**Discussion:** Let
\[
i : \text{Hom}_R(\text{Res}_R^S M, N) \to \text{Hom}_S(M, \text{Hom}_R(S,N))
\]
be defined for \( \varphi \in \text{Hom}_R(\text{Res}_R^S M, N) \) by
\[
i(\varphi)(m)(s) = \varphi(s \cdot m)
\]
This makes some sense, at least, since \( M \) is an \( S \)-module. We must verify that \( i(\varphi) : M \to \text{Hom}_R(S,N) \) is \( S \)-linear. Note that the \( S \)-module structure on \( \text{Hom}_R(S,N) \) is
\[
(s \cdot \psi)(t) = \psi(st)
\]
where \( s,t \in S \), \( \psi \in \text{Hom}_R(S,N) \). Then we check:
\[
(i(\varphi)(sm))(t) = i(\varphi)(t \cdot sm) = i(\varphi)(stm) = i(\varphi)(m)(st) = (s \cdot i(\varphi)(m))(t)
\]
which proves the \( S \)-linearity.

The map \( j \) in the other direction is described, for \( \Phi \in \text{Hom}_S(M, \text{Hom}_R(S,N)) \), by
\[
j(\Phi)(m) = \Phi(m)(1_S)
\]
where \( 1_S \) is the identity in \( S \). Verify that these are mutual inverses, by
\[
i(j(\Phi))(m)(s) = j(\Phi)(s \cdot m) = \Phi(sm)(1_S) = (s \cdot \Phi(m))(1_S) = \Phi(m)(s \cdot 1_S) = \Phi(m)(s)
\]
as hoped. (Again, the equality
\[
(s \cdot \Phi(m))(1_S) = \Phi(m)(s \cdot 1_S)
\]
is the definition of the \( S \)-module structure on \( \text{Hom}_R(S,N) \).) In the other direction,
\[
j(i(\varphi))(m) = i(\varphi)(m)(1_S) = \varphi(1 \cdot m) = \varphi(m)
\]
Thus, \( i \) and \( j \) are mutual inverses, so are isomorphisms.

For naturality, let \( f : M \to M' \) be an \( S \)-module homomorphism. Add indices to the previous notation, so that
\[
i_{M,N} : \text{Hom}_R(\text{Res}_R^S M, N) \to \text{Hom}_S(M, \text{Hom}_R(S,N))
\]
is the isomorphism discussed just above, and $i_{M',N}$ the analogous isomorphism for $M'$ and $N$. We must show that the diagram

$$\begin{align*}
\Hom_R(\Res_S^R M, N) & \xrightarrow{i_{M,N}} \Hom_S(M, \Hom_R(S, N)) \\
\Hom_R(\Res_S^R M', N) & \xrightarrow{i_{M',N}} \Hom_S(M', \Hom_R(S, N)) \quad \text{commutes, where } - \circ f \text{ is pre-composition with } f. \end{align*}$$

(We use the same symbol for the map $f : M \to M'$ on the modules whose $S$-structure has been forgotten, leaving only the $R$-module structure.) Starting in the lower left of the diagram, going up then right, for $\varphi \in \Hom_R(\Res_S^R M', N)$,

$$(i_{M,N} \circ (- \circ f) \varphi) (m)(s) = (i_{M,N}(\varphi \circ f)) (m)(s) = (\varphi \circ f)(s \cdot m) = \varphi(f(s \cdot m))$$

On the other hand, going right, then up,

$$((- \circ f) \circ i_{M',N} \varphi) (m)(s) = (i_{M',N} \varphi)(fm)(s) = \varphi(s \cdot fm) = \varphi(f(s \cdot m))$$

since $f$ is $S$-linear. That is, the two outcomes are the same, so the diagram commutes, proving functoriality in $M$, which is a part of the naturality assertion. ///

[08.58] Let

$$M = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \quad N = \mathbb{Z} \oplus 4\mathbb{Z} \oplus 24\mathbb{Z} \oplus 144\mathbb{Z}$$

What are the elementary divisors of $\bigwedge^2(M/N)$?

**Discussion:** First, note that this is *not* the same as asking about the structure of $(\bigwedge^2 M)/(\bigwedge^2 N)$. Still, we can address that, too, after dealing with the question that was asked.

First,

$$M/N = \mathbb{Z}/4 \oplus \mathbb{Z}/24 \oplus \mathbb{Z}/144 \approx \mathbb{Z}/4 \oplus \mathbb{Z}/24 \oplus \mathbb{Z}/144$$

where we use the obvious slightly lighter notation. Generators for $M/N$ are

$$m_1 = 1 \oplus 0 \oplus 0 \quad m_2 = 0 \oplus 1 \oplus 0 \quad m_3 = 0 \oplus 0 \oplus 1$$

where the 1s are respectively in $\mathbb{Z}/4$, $\mathbb{Z}/24$, and $\mathbb{Z}/144$. We know that $e_i \wedge e_j$ *generate* the exterior square, for the 3 pairs of indices with $i < j$. Much as in the computation of $\mathbb{Z}/a \otimes \mathbb{Z}/b$, for $e$ in a $\mathbb{Z}$-module $E$ with $a \cdot e = 0$ and $f$ in $E$ with $b \cdot f = 0$, let $r,s$ be integers such that $$ra + sb = \gcd(a,b)$$

Then $$\gcd(a,b) \cdot e \wedge f = r(\gcd(a,b) \wedge f) = r \cdot 0 + s \cdot 0 = 0$$

Thus, $4 \cdot e_1 \wedge e_2 = 0$ and $4 \cdot e_1 \wedge e_3 = 0$, while $24 \cdot e_2 \wedge e_3 = 0$. If there are no further relations, then we could have

$$\bigwedge^2(M/N) \approx \mathbb{Z}/4 \oplus \mathbb{Z}/24 \oplus \mathbb{Z}/24$$

(so the elementary divisors would be 4, 4, 24.)

To prove, in effect, that there are no further relations than those just indicated, we must construct suitable alternating bilinear maps. Suppose for $r,s,t \in \mathbb{Z}$

$$r \cdot e_1 \wedge e_2 + s \cdot e_1 \wedge e_3 + t \cdot e_2 \wedge e_3 = 0$$
Let
\[ B_{12} : (\mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}e_3) \times (\mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}e_3) \rightarrow \mathbb{Z}/4 \]
by
\[ B_{12}(xe_1 + ye_2 + ze_3, \xi e_1 + \eta e_2 + \zeta e_3) = (x\eta - \xi y) + 4\mathbb{Z} \]
(As in earlier examples, since 4|4 and 4|24, this is well-defined.) By arrangement, this \( B_{12} \) is alternating, and induces a unique linear map \( \beta_{12} \) on \( \bigwedge^2 (M/N) \), with
\[ \beta_{12}(e_1 \wedge e_2) = 1 \quad \beta_{12}(e_1 \wedge e_3) = 0 \quad \beta_{12}(e_2 \wedge e_3) = 0 \]
Applying this to the alleged relation, we find that \( r \equiv 0 \mod 4 \). Similar constructions for the other two pairs of indices \( i < j \) show that \( s \equiv 0 \mod 4 \) and \( t \equiv 0 \mod 24 \). This shows that we have all the relations, and
\[ \bigwedge^2 (M/N) \approx \mathbb{Z}/4 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/24 \]
as hoped/claimed. ///

**Now consider the other version of this question.** Namely, letting
\[ M = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \quad N = \mathbb{Z} \oplus 4\mathbb{Z} \oplus 24\mathbb{Z} \oplus 144\mathbb{Z} \]
compute the elementary divisors of \( (\bigwedge^2 M)/(\bigwedge^2 N) \).
Let \( e_1, e_2, e_3, e_4 \) be the standard basis for \( \mathbb{Z}^4 \). Let \( i : N \rightarrow M \) be the inclusion. We have shown that exterior powers of free modules are free with the expected generators, so \( M \) is free on
\[ e_1 \wedge e_2, \quad e_1 \wedge e_3, \quad e_1 \wedge e_4, \quad e_2 \wedge e_3, \quad e_2 \wedge e_4, \quad e_3 \wedge e_4 \]
and \( N \) is free on
\[ (1 \cdot 4) e_1 \wedge e_2, \quad (1 \cdot 24) e_1 \wedge e_3, \quad (1 \cdot 144) e_1 \wedge e_4, \quad (4 \cdot 24) e_2 \wedge e_3, \quad (4 \cdot 144) e_2 \wedge e_4, \quad (24 \cdot 144) e_3 \wedge e_4 \]
The inclusion \( i : N \rightarrow M \) induces a natural map \( \bigwedge^2 i : \bigwedge^2 \rightarrow \bigwedge^2 M \), taking \( r \cdot e_i \wedge e_j \) (in \( N \)) to \( r \cdot e_i \wedge e_j \) (in \( M \)). Thus, the quotient of \( \bigwedge^2 M \) by (the image of) \( \bigwedge^2 N \) is visibly
\[ \mathbb{Z}/4 \oplus \mathbb{Z}/24 \oplus \mathbb{Z}/144 \oplus \mathbb{Z}/96 \oplus \mathbb{Z}/576 \oplus \mathbb{Z}/3456 \]
The integers 4, 24, 144, 96, 576, 3456 do not quite have the property 4|24|144|96|576|3456, so are not elementary divisors. The problem is that neither 144|96 nor 96|144. The only primes dividing all these integers are 2 and 3, and, in particular,
\[ 4 = 2^2, \quad 24 = 2^3 \cdot 3, \quad 144 = 2^4 \cdot 3^2, \quad 96 = 2^5 \cdot 3, \quad 576 = 2^6 \cdot 3^2, \quad 3456 = 2^7 \cdot 3^3, \]
From Sun-Ze’s theorem,
\[ \mathbb{Z}/(2^a \cdot 3^b) \approx \mathbb{Z}/2^a \oplus \mathbb{Z}/3^b \]
so we can rewrite the summands \( \mathbb{Z}/144 \) and \( \mathbb{Z}/96 \) as
\[ \mathbb{Z}/144 \oplus \mathbb{Z}/96 \approx (\mathbb{Z}/2^4 \oplus \mathbb{Z}/3^2) \oplus (\mathbb{Z}/2^5 \oplus \mathbb{Z}/3) \approx (\mathbb{Z}/2^4 \oplus \mathbb{Z}/3) \oplus (\mathbb{Z}/2^5 \oplus \mathbb{Z}/3^2) \approx \mathbb{Z}/48 \oplus \mathbb{Z}/288 \]
Now we do have 4|24|48|288|576|3456, and
\[ (\bigwedge^2 M)/(\bigwedge^2 N) \approx \mathbb{Z}/4 \oplus \mathbb{Z}/24 \oplus \mathbb{Z}/48 \oplus \mathbb{Z}/288 \oplus \mathbb{Z}/576 \oplus \mathbb{Z}/3456 \]
is in elementary divisor form. ///