[02.1] Find all the idempotent elements in \( \mathbb{Z}[i]/(13) \).

[02.2] Find all the nilpotent elements in \( \mathbb{Z}[i]/(2) \).

[02.3] (Lagrange interpolation) Let \( \alpha_1, \ldots, \alpha_n \) be distinct elements in a field \( k \), and let \( \beta_1, \ldots, \beta_n \) be any elements of \( k \). Prove that there is a unique polynomial \( P(x) \) of degree \( < n \) in \( k[x] \) such that, for all indices \( i \),

\[
P(\alpha_i) = \beta_i
\]

Indeed, letting

\[
Q(x) = \prod_{i=1}^{n} (x - \alpha_i)
\]

show that

\[
P(x) = \sum_{i=1}^{n} \frac{Q(x)}{(x - \alpha_i) \cdot Q'(\alpha_i)} \cdot \beta_i
\]

[02.4] (Simple case of partial fractions) Let \( \alpha_1, \ldots, \alpha_n \) be distinct elements in a field \( k \). Let \( R(x) \) be any polynomial in \( k[x] \) of degree \( < n \). Show that there exist unique constants \( c_i \in k \) such that in the field of rational functions \( k(x) \)

\[
\frac{R(x)}{(x - \alpha_1) \cdots (x - \alpha_n)} = \frac{c_1}{x - \alpha_1} + \cdots + \frac{c_n}{x - \alpha_n}
\]

In particular, let

\[
Q(x) = \prod_{i=1}^{n} (x - \alpha_i)
\]

and show that

\[
c_i = \frac{R(\alpha_i)}{Q'(\alpha_i)}
\]

[02.5] (Analogue of partial fractions for rational numbers) Show that every positive rational number is expressible as

\[
\ell + \sum_p \frac{c_p}{p^{n_p}} \quad (0 \leq \ell \in \mathbb{Z}, \text{ distinct primes } p, \text{ exponents } 0 \leq n_p \in \mathbb{Z}, \text{ integers } 0 \leq c_p < p^{n_p})
\]

[02.6] Show that the ideal \( I \) generated in \( \mathbb{Z}[x] \) by \( x^2 + 1 \) and 5 is not maximal.

[02.7] Show that the ideal \( I \) generated in \( \mathbb{Z}[x] \) by \( x^2 + x + 1 \) and 11 is maximal.

[02.8] Let \( k \) be a field. Given \( P \in k[x] \) of degree \( n \), show that there is a \( k \)-linear map \( T : k^n \to k^n \) such that \( P(T) = 0 \).

[02.9] Determine all two-sided ideals in the ring of \( n \)-by-\( n \) matrices with entries in a field \( k \).

[02.10] Let \( V_1 \subset \ldots \subset V_{n-1} \subset V \) and \( W_1 \subset \ldots \subset W_{n-1} \subset V \) be two maximal flags in an \( n \)-dimensional vector space \( V \) over a field \( k \). Show that there is a \( k \)-linear map \( T : V \to V \) such that \( TV_i = W_i \).