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Discussion 04

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[04.1] Given a 3-by-3 matrix M with integer entries, find A, B integer 3-by-3 matrices with determinant ± 1 such that AMB is diagonal.

[04.2] Given a row vector $x = (x_1, \ldots, x_n)$ of integers whose *gcd* is 1, prove that there exists an n-by-n integer matrix M with determinant ± 1 such that $xM = (0, \ldots, 0, 1)$.

[04.3] Given a row vector $x = (x_1, \ldots, x_n)$ of integers whose *gcd* is 1, prove that there exists an n-by-n integer matrix M with determinant ± 1 whose bottom row is x.

[04.4] Show that $GL(2, \mathbb{F}_2)$ is isomorphic to the permutation group S_3 on three letters.

[04.5] Determine all conjugacy classes in $GL(2, \mathbb{F}_3)$.

[04.6] Determine all conjugacy classes in $GL(3, \mathbb{F}_2)$.

[04.7] Determine all conjugacy classes in $GL(4, \mathbb{F}_2)$.

[04.8] Tell a p-Sylow subgroup in $GL(3, \mathbb{F}_p)$.

[04.9] Tell a 3-Sylow subgroup in $GL(3, \mathbb{F}_7)$.

[04.10] Tell a 19-Sylow subgroup in $GL(3, \mathbb{F}_7)$.

[04.11] Classify the conjugacy classes in S_n (the *symmetric group* of bijections of $\{1, \ldots, n\}$ to itself).

[04.12] The **projective linear group** $PGL_n(k)$ is the group $GL_n(k)$ modulo its center k, which is the collection of scalar matrices. Prove that $PGL_2(\mathbb{F}_3)$ is isomorphic to S_4, the group of permutations of 4 things. (*Hint:* Let $PGL_2(\mathbb{F}_3)$ act on **lines** in $\mathbb{F}_3^2$, that is, on one-dimensional $\mathbb{F}_3$-subspaces in $\mathbb{F}_3^2$.)

[04.13] An automorphism of a group G is **inner** if it is of the form $g \to xgx^{-1}$ for fixed $x \in G$. Otherwise it is an **outer automorphism**. Show that every automorphism of the permutation group S_3 on 3 things is *inner*. (*Hint:* Compare the action of S_3 on the set of 2-cycles by conjugation.)

[04.14] Identify the element of S_n requiring the maximal number of adjacent transpositions to express it, and prove that it is unique.

[04.15] Let the permutation group S_n on n things act on the polynomial ring $\mathbb{Z}[x_1, \ldots, x_n]$ by $p(x_i) = x_{p(i)}$ for $p \in S_n$. Verify that this is a group homomorphism

$$S_n \longrightarrow \mathrm{Aut}_{\mathbb{Z}-\mathrm{alg}}(\mathbb{Z}[x_1, \ldots, x_n])$$

Consider

$$D = \prod_{i<j} (x_i - x_j)$$

Show that for any $p \in S_n$

$$p(D) = \sigma(p) \cdot D$$

where $\sigma(p) = \pm 1$. Infer that σ is a (non-trivial) group homomorphism, the **sign** homomorphism on S_n.
