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Examples 05

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[05.1] Classify the *conjugacy classes* in S_n (the *symmetric group* of bijections of $\{1, \ldots, n\}$ to itself).

[05.2] The **projective linear group** $PGL_n(k)$ is the group $GL_n(k)$ modulo its center k, which is the collection of scalar matrices. Prove that $PGL_2(\mathbb{F}_3)$ is isomorphic to S_4, the group of permutations of 4 things. (*Hint:* Let $PGL_2(\mathbb{F}_3)$ act on **lines** in $\mathbb{F}_3^2$, that is, on one-dimensional $\mathbb{F}_3$-subspaces in $\mathbb{F}_3^2$.)

[05.3] An automorphism of a group G is **inner** if it is of the form $g \to xgx^{-1}$ for fixed $x \in G$. Otherwise it is an **outer automorphism**. Show that every automorphism of the permutation group S_3 on 3 things is *inner*. (*Hint:* Compare the action of S_3 on the set of 2-cycles by conjugation.)

[05.4] Identify the element of S_n requiring the maximal number of adjacent transpositions to express it, and prove that it is unique.

[05.5] Let the permutation group S_n on n things act on the polynomial ring $\mathbb{Z}[x_1, \ldots, x_n]$ by $\mathbb{Z}$-algebra homomorphisms defined by $p(x_i) = x_{p(i)}$ for $p \in S_n$. (The universal mapping property of the polynomial ring allows us to define the images of the indeterminates x_i to be whatever we want, and at the same time guarantees that this determines the $\mathbb{Z}$-algebra homomorphism completely.) Verify that this is a group homomorphism $S_n \to \mathrm{Aut}_{\mathbb{Z}-\mathrm{alg}}(\mathbb{Z}[x_1, \ldots, x_n])$. Consider $D = \prod_{i<j} (x_i - x_j)$. Show that, for any $p \in S_n$, $p(D) = \sigma(p) \cdot D$, where $\sigma(p) = \pm 1$. Infer that σ is a (non-trivial) group homomorphism, the **sign** homomorphism on S_n.

[05.6] Let R be a principal ideal domain. Let I be a non-zero prime ideal in R. Show that I is *maximal*.

[05.7] Let k be a field. Show that in the polynomial ring $k[x, y]$ in two variables the ideal $I = k[x, y] \cdot x + k[x, y] \cdot y$ is *not* principal.

[05.8] Let k be a field, and let $R = k[x_1, \ldots, x_n]$. Show that the inclusions of ideals

$$Rx_1 \subset Rx_1 + Rx_2 \subset \ldots \subset Rx_1 + \ldots + Rx_n$$

are *strict*, and that all these ideals are *prime*.

[05.9] Let k be a field. Show that the ideal M generated by $x_1, \ldots, x_n$ in the polynomial ring $R = k[x_1, \ldots, x_n]$ is *maximal* (proper).

[05.10] Show that the maximal ideals in $R = \mathbb{Z}[x]$ are all of the form $R \cdot p + R \cdot f(x)$, where p is a prime and $f(x)$ is a monic polynomial which is irreducible modulo p.

[05.11] For x, y non-zero elements of a PID R be a PID, determine $\mathrm{Hom}_R(R/\langle x\rangle, R/\langle y\rangle)$.

[05.12] *(A warm-up to Hensel's lemma)* Let p be an odd prime. Fix $a \not\equiv 0 \bmod p$ and suppose $x^2 = a \bmod p$ has a solution x_1. Show that for every positive integer n the congruence $x^2 = a \bmod p^n$ has a solution x_n. (*Hint:* Try $x_{n+1} = x_n + p^n y$ and solve for y mod p).

[05.13] *(Another warm-up to Hensel's lemma)* Let p be a prime not 3. Fix $a \neq 0 \bmod p$ and suppose $x^3 = a \bmod p$ has a solution x_1. Show that for every positive integer n the congruence $x^3 = a \bmod p^n$ has a solution x_n. (*Hint:* Try $x_{n+1} = x_n + p^n y$ and solve for y mod p).]
