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Discussion 06

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[06.1] Show that a *finite* integral domain (no zero divisors) is necessarily a *field*.

[06.2] Let $P(x) = x^3 + ax + b \in k[x]$. Suppose that $P(x)$ factors into linear polynomials $P(x) = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3)$. Give a polynomial condition on a, b for the α_i to be distinct.

[06.3] The first three **elementary symmetric functions** in indeterminates $x_1, \ldots, x_n$ are

$$\sigma_1 = \sigma_1(x_1, \ldots, x_n) = x_1 + x_2 + \ldots + x_n = \sum_i x_i$$

$$\sigma_2 = \sigma_2(x_1, \ldots, x_n) = \sum_{i<j} x_i x_j$$

$$\sigma_3 = \sigma_3(x_1, \ldots, x_n) = \sum_{i<j<\ell} x_i x_j x_\ell$$

Express $x_1^3 + x_2^3 + \ldots + x_n^3$ in terms of $\sigma_1, \sigma_2, \sigma_3$.

[06.4] Express $\sum_{i \neq j} x_i^2 x_j$ as a polynomial in the elementary symmetric functions of $x_1, \ldots, x_n$.

[06.5] Suppose the characteristic of the field k does not divide n. Let $\ell > 2$. Show that

$$P(x_1, \ldots, x_n) = x_1^n + \ldots + x_\ell^n$$

is irreducible in $k[x_1, \ldots, x_\ell]$.

[06.6] Find the determinant of the **circulant** matrix

$$\begin{pmatrix} x_1 & x_2 & \ldots & x_{n-2} & x_{n-1} & x_n \\ x_n & x_1 & x_2 & \ldots & x_{n-2} & x_{n-1} \\ x_{n-1} & x_n & x_1 & x_2 & \ldots & x_{n-2} \\ \vdots & & & \ddots & & \vdots \\ x_3 & & & & x_1 & x_2 \\ x_2 & x_3 & \ldots & & x_n & x_1 \end{pmatrix}$$

(*Hint:* Let ζ be an n^{th} root of 1. If $x_{i+1} = \zeta \cdot x_i$ for all indices $i < n$, then the $(j+1)^{th}$ row is ζ times the j^{th}, and the determinant is 0.)

[06.7] Let $f(x)$ be a monic polynomial with integer coefficients. Show that f is irreducible in $\mathbb{Q}[x]$ if it is irreducible in $(\mathbb{Z}/p)[x]$ for some p.

[06.8] Let n be a positive integer such that $(\mathbb{Z}/n)^\times$ is *not* cyclic. Show that the n^{th} cyclotomic polynomial $\Phi_n(x)$ factors properly in $\mathbb{F}_p[x]$ for any prime p not dividing n.

[06.9] Show that the 15^{th} cyclotomic polynomial $\Phi_{15}(x)$ is irreducible in $\mathbb{Q}[x]$, despite being reducible in $\mathbb{F}_p[x]$ for every prime p.

[06.10] Let p be a prime. Show that every degree d irreducible in $\mathbb{F}_p[x]$ is a factor of $x^{p^d-1} - 1$. Show that that the $(p^d - 1)^{th}$ cyclotomic polynomial's irreducible factors in $\mathbb{F}_p[x]$ are all of degree d.

[06.11] Fix a prime p, and let ζ be a primitive p^{th} root of 1 (that is, $\zeta^p = 1$ and no smaller exponent will do). Let

$$V = \det \begin{pmatrix} 1 & 1 & 1 & 1 & \ldots & 1 \\ 1 & \zeta & \zeta^2 & \zeta^3 & \ldots & \zeta^{p-1} \\ 1 & \zeta^2 & (\zeta^2)^2 & (\zeta^2)^3 & \ldots & (\zeta^2)^{p-1} \\ 1 & \zeta^3 & (\zeta^3)^2 & (\zeta^3)^3 & \ldots & (\zeta^3)^{p-1} \\ 1 & \zeta^4 & (\zeta^4)^2 & (\zeta^4)^3 & \ldots & (\zeta^4)^{p-1} \\ \vdots & & & & & \vdots \\ 1 & \zeta^{p-1} & (\zeta^{p-1})^2 & (\zeta^{p-1})^3 & \ldots & (\zeta^{p-1})^{p-1} \end{pmatrix}$$

Compute the rational number V^2.

[06.12] Let $K = \mathbb{Q}(\zeta)$ where ζ is a primitive 15^{th} root of unity. Find 4 fields k strictly between $\mathbb{Q}$ and K.

[06.13] Let ζ be a primitive n^{th} root of unity in a field of characteristic 0. Let M be the n-by-n matrix with ij^{th} entry ζ^{ij}. Find the multiplicative inverse of M.

[06.14] Let $\mu = \alpha\beta^2 + \beta\gamma^2 + \gamma\alpha^2$ and $\nu = \alpha^2\beta + \beta^2\gamma + \gamma^2\alpha$. Show that these are the two roots of a quadratic equation with coefficients in $\mathbb{Z}[s_1, s_2, s_3]$ where the s_i are the elementary symmetric polynomials in α, β, γ.

[06.15] The 5^{th} cyclotomic polynomial $\Phi_5(x)$ factors into two irreducible quadratic factors over $\mathbb{Q}(\sqrt{5})$. Find the two irreducible factors.

[06.16] The 7^{th} cyclotomic polynomial $\Phi_7(x)$ factors into two irreducible cubic factors over $\mathbb{Q}(\sqrt{-7})$. Find the two irreducible factors.

[06.17] Let ζ be a primitive 13^{th} root of unity in an algebraic closure of $\mathbb{Q}$. Find an element α in $\mathbb{Q}(\zeta)$ which satisfies an irreducible cubic with rational coefficients. Find an element β in $\mathbb{Q}(\zeta)$ which satisfies an irreducible quartic with rational coefficients. Determine the cubic and the quartic explicitly.

[06.18] Let $f(x) = x^8 + x^6 + x^4 + x^2 + 1$. Show that f factors into two irreducible quartics in $\mathbb{Q}[x]$. Show that

$$x^8 + 5x^6 + 25x^4 + 125x^2 + 625$$

also factors into two irreducible quartics in $\mathbb{Q}[x]$.
