[06.1] Show that a finite integral domain (no zero divisors) is necessarily a field.

[06.2] Let $P(x) = x^3 + ax + b \in k[x]$. Suppose that $P(x)$ factors into linear polynomials $P(x) = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3)$. Give a polynomial condition on $a, b$ for the $\alpha_i$ to be distinct.

[06.3] The first three elementary symmetric functions in indeterminates $x_1, \ldots, x_n$ are

\[
\begin{align*}
\sigma_1 &= \sigma_1(x_1, \ldots, x_n) = x_1 + x_2 + \ldots + x_n = \sum_i x_i \\
\sigma_2 &= \sigma_2(x_1, \ldots, x_n) = \sum_{i<j} x_ix_j \\
\sigma_3 &= \sigma_3(x_1, \ldots, x_n) = \sum_{i<j<\ell} x_ix_jx_\ell
\end{align*}
\]

Express $x_1^3 + x_2^3 + \ldots + x_n^3$ in terms of $\sigma_1, \sigma_2, \sigma_3$.

[06.4] Express $\sum_{i\neq j} x_i^2x_j$ as a polynomial in the elementary symmetric functions of $x_1, \ldots, x_n$.

[06.5] Suppose the characteristic of the field $k$ does not divide $n$. Let $\ell > 2$. Show that $P(x_1, \ldots, x_n) = x_1^n + \ldots + x_\ell^n$ is irreducible in $k[x_1, \ldots, x_\ell]$.

[06.6] Find the determinant of the circulant matrix

\[
\begin{pmatrix}
 x_1 & x_2 & \cdots & x_{n-2} & x_{n-1} & x_n \\
 x_n & x_1 & \cdots & x_{n-2} & x_{n-1} & x_2 \\
 x_{n-1} & x_n & x_1 & \cdots & x_{n-2} & \vdots \\
 \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
 x_3 & x_2 & x_1 & x_2 & x_3 & \vdots \\
 x_2 & x_3 & \cdots & x_n & x_1 & x_2
\end{pmatrix}
\]

(Hint: Let $\zeta$ be an $n^{th}$ root of 1. If $x_{i+1} = \zeta \cdot x_i$ for all indices $i < n$, then the $(j+1)^{th}$ row is $\zeta$ times the $j^{th}$, and the determinant is 0.)

[06.7] Let $f(x)$ be a monic polynomial with integer coefficients. Show that $f$ is irreducible in $\mathbb{Q}[x]$ if it is irreducible in $(\mathbb{Z}/p)[x]$ for some $p$.

[06.8] Let $n$ be a positive integer such that $(\mathbb{Z}/n)^\times$ is not cyclic. Show that the $n^{th}$ cyclotomic polynomial $\Phi_n(x)$ factors properly in $\mathbb{F}_p[x]$ for any prime $p$ not dividing $n$.

[06.9] Show that the $15^{th}$ cyclotomic polynomial $\Phi_{15}(x)$ is irreducible in $\mathbb{Q}[x]$, despite being reducible in $\mathbb{F}_p[x]$ for every prime $p$.

[06.10] Let $p$ be a prime. Show that every degree $d$ irreducible in $\mathbb{F}_p[x]$ is a factor of $x^{p^d-1} - 1$. Show that the $(p^d - 1)^{th}$ cyclotomic polynomial’s irreducible factors in $\mathbb{F}_p[x]$ are all of degree $d$. 
[06.11] Fix a prime $p$, and let $\zeta$ be a primitive $p^{th}$ root of 1 (that is, $\zeta^p = 1$ and no smaller exponent will do). Let

$$V = \det \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \zeta & \zeta^2 & \zeta^3 & \cdots & \zeta^{p-1} \\ 1 & \zeta^2 & (\zeta^2)^2 & (\zeta^2)^3 & \cdots & (\zeta^2)^{p-1} \\ 1 & \zeta^3 & (\zeta^3)^2 & (\zeta^3)^3 & \cdots & (\zeta^3)^{p-1} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \zeta^{p-1} & (\zeta^{p-1})^2 & (\zeta^{p-1})^3 & \cdots & (\zeta^{p-1})^{p-1} \end{pmatrix}$$

Compute the rational number $V^2$.

[06.12] Let $K = \mathbb{Q}(\zeta)$ where $\zeta$ is a primitive $15^{th}$ root of unity. Find 4 fields $k$ strictly between $\mathbb{Q}$ and $K$.

[06.13] Let $\zeta$ be a primitive $n^{th}$ root of unity in a field of characteristic 0. Let $M$ be the $n$-by-$n$ matrix with $i^j$ entry $\zeta^i$. Find the multiplicative inverse of $M$.

[06.14] Let $\mu = \alpha \beta^2 + \beta \gamma^2 + \gamma \alpha^2$ and $\nu = \alpha^2 \beta + \beta^2 \gamma + \gamma^2 \alpha$. Show that these are the two roots of a quadratic equation with coefficients in $\mathbb{Z}[s_1, s_2, s_3]$ where the $s_i$ are the elementary symmetric polynomials in $\alpha, \beta, \gamma$.

[06.15] The $5^{th}$ cyclotomic polynomial $\Phi_5(x)$ factors into two irreducible quadratic factors over $\mathbb{Q}(\sqrt{5})$. Find the two irreducible factors.

[06.16] The $7^{th}$ cyclotomic polynomial $\Phi_7(x)$ factors into two irreducible cubic factors over $\mathbb{Q}(\sqrt{-7})$. Find the two irreducible factors.

[06.17] Let $\zeta$ be a primitive $13^{th}$ root of unity in an algebraic closure of $\mathbb{Q}$. Find an element $\alpha$ in $\mathbb{Q}(\zeta)$ which satisfies an irreducible cubic with rational coefficients. Find an element $\beta$ in $\mathbb{Q}(\zeta)$ which satisfies an irreducible quartic with rational coefficients. Determine the cubic and the quartic explicitly.

[06.18] Let $f(x) = x^8 + x^6 + x^4 + x^2 + 1$. Show that $f$ factors into two irreducible quartics in $\mathbb{Q}[x]$. Show that

$$x^8 + 5x^6 + 25x^4 + 125x^2 + 625$$

also factors into two irreducible quartics in $\mathbb{Q}[x]$.