[07.1] Let $k$ be a field of characteristic 0. Let $f$ be an irreducible polynomial in $k[x]$. Prove that $f$ has no repeated factors, even over an algebraic closure of $k$.

[07.2] Let $K$ be a finite extension of a field $k$ of characteristic 0. Prove that $K$ is separable over $k$.

[07.3] Let $k$ be a field of characteristic $p > 0$. Suppose that $k$ is **perfect**, meaning that for any $a \in k$ there exists $b \in k$ such that $b^p = a$. Let $f(x) = \sum c_i x^i$ in $k[x]$ be a polynomial such that its (algebraic) derivative

$$f'(x) = \sum c_i i x^{i-1}$$

is the zero polynomial. Show that there is a unique polynomial $g \in k[x]$ such that $f(x) = g(x)^p$.

[07.4] Let $k$ be a perfect field of characteristic $p > 0$, and $f$ an irreducible polynomial in $k[x]$. Show that $f$ has no repeated factors (even over an algebraic closure of $k$).

[07.5] Show that all finite fields $\mathbb{F}_{p^n}$ with $p$ prime and $1 \leq n \in \mathbb{Z}$ are perfect.

[07.6] Let $K$ be a finite extension of a finite field $k$. Prove that $K$ is separable over $k$.

[07.7] Find all fields intermediate between $\mathbb{Q}$ and $\mathbb{Q}(\zeta)$ where $\zeta$ is a primitive $17^{th}$ root of unity.

[07.8] Let $f, g$ be relatively prime polynomials in $n$ indeterminates $t_1, \ldots, t_n$, with $g$ not 0. Suppose that the ratio $f(t_1, \ldots, t_n)/g(t_1, \ldots, t_n)$ is invariant under all permutations of the $t_i$. Show that both $f$ and $g$ are polynomials in the elementary symmetric functions in the $t_i$. 