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Examples 08

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[08.1] Let $T \in \mathrm{Hom}_k(V)$ for a finite-dimensional k-vectorspace V, with k a field. Let W be a T-stable subspace. Prove that the minimal polynomial of T on W is a divisor of the minimal polynomial of T on V. Define a natural action of T on the quotient V/W, and prove that the minimal polynomial of T on V/W is a divisor of the minimal polynomial of T on V.

[08.2] Let $T \in \mathrm{Hom}_k(V)$ for a finite-dimensional k-vectorspace V, with k a field. Suppose that T is *diagonalizable* on V. Let W be a T-stable subspace of V. Show that T is diagonalizable on W.

[08.3] Let $T \in \mathrm{Hom}_k(V)$ for a finite-dimensional k-vectorspace V, with k a field. Suppose that T is *diagonalizable* on V, with *distinct eigenvalues*. Let $S \in \mathrm{Hom}_k(V)$ commute with T, in the natural sense that $ST = TS$. Show that S is diagonalizable on V.

[08.4] Let $T \in \mathrm{Hom}_k(V)$ for a finite-dimensional k-vectorspace V, with k a field. Suppose that T is *diagonalizable* on V. Show that $k[T]$ contains the projectors to the eigenspaces of T.

[08.5] Let V be a complex vector space with a (positive definite) inner product. Show that $T \in \mathrm{Hom}_k(V)$ cannot be a normal operator if it has any non-trivial Jordan block.

[08.6] Show that a positive-definite hermitian n-by-n matrix A has a unique positive-definite square root B (that is, $B^2 = A$).

[08.7] Given a square n-by-n complex matrix M, show that there are unitary matrices A and B such that AMB is *diagonal*.

[08.8] Given a square n-by-n complex matrix M, show that there is a unitary matrix A such that AM is *upper triangular*.

[08.9] Let Z be an m-by-n complex matrix. Let Z^* be its conjugate-transpose. Show that

$$\det(1_m - ZZ^*) = \det(1_n - Z^*Z)$$

[08.10] Give an example of two commuting diagonalizable operators S, T on a 4-dimensional vectorspace V over a field k such that each operator has exactly two eigenvalues, and the eigenspaces are two-dimensional, but/and the intersection of any S-eigenspace with any T-eigenspace is just 1-dimensional. Explain why this does not contradict results about simultaneous eigenvectors

[08.11] Let T be a diagonalizable operator on a finite-dimensional vector space V over a field k. Suppose that some T-eigenspace is not one-dimensional. Exhibit a diagonalizable endomorphism S of V commuting with T *not* lying in $k[T]$.

[08.12] Let $\lambda_1, \ldots, \lambda_n$ be distinct elements of a field k. Let $\mu_1, \ldots, \mu_n$ be arbitrary elements of k. Show that there is a unique polynomial $f(x)$ in $k[x]$ of degree $\le n-1$ such that $f(\lambda_i) = \mu_i$ for all i.

[08.13] Let T be a diagonalizable operator on a finite-dimensional vector space V over a field k. Suppose that all the eigenspaces are one-dimensional. Prove that any endomorphism commuting with T is in $k[T]$.

[08.14] Let S, T be commuting diagonalizable endomorphisms of a finite-dimensional vector space V over a field k. Suppose that there is a basis $\{v_1, \ldots, v_n\}$ of simultaneous eigenvectors such that for $i \neq j$ the two vectors v_i and v_j either have different eigenvalues for S or have different eigenvalues for T. Show that there is a single diagonalizable operator R on V such that $k[S,T] = k[R]$.

[08.15] Give an example of a diagonalizable operator T on a 2-dimensional complex vector space V (with hermitian inner product $\langle , \rangle$) with eigenvectors v, w such that application of the Gram-Schmidt process does *not* yield two orthonormal *eigenvectors*.

[08.16] Let S be a hermitian operator on a finite-dimensional complex vector space V with hermitian inner product $\langle , \rangle$. Let W be a S-stable subspace of V. Show that S is hermitian on W.

[08.17] Let S, T be commuting hermitian operators on a finite-dimensional complex vector space V with hermitian inner product $\langle , \rangle$. Show that there is an orthonormal basis for V consisting of simultaneous eigenvectors for both S and T.

[08.18] Let k be a field, and V a finite-dimensional k vectorspace. Let Λ be a subset of the dual space V^*, with $|\Lambda| < \dim V$. Show that the **homogeneous system of equations**

$$\lambda(v) = 0 \quad (\text{for all } \lambda \in \Lambda)$$

has a non-trivial (that is, non-zero) solution $v \in V$ (meeting all these conditions).

[08.19] Let k be a field, and V a finite-dimensional k vectorspace. Let Λ be a *linearly independent* subset of the dual space V^*. Let $\lambda \to a_\lambda$ be a set map $\Lambda \to k$. Show that an **inhomogeneous system of equations**

$$\lambda(v) = a_\lambda \quad (\text{for all } \lambda \in \Lambda)$$

has a solution $v \in V$ (meeting all these conditions).

[08.20] Let T be a k-linear endomorphism of a finite-dimensional k-vectorspace V. For an eigenvalue λ of T, let V_λ be the generalized λ-eigenspace

$$V_\lambda = \{v \in V : (T - \lambda)^n v = 0 \text{ for some } 1 \leq n \in \mathbb{Z}\}$$

Show that the projector P of V to V_λ (commuting with T) lies inside $k[T]$.

[08.21] Let T be a matrix in Jordan normal form with entries in a field k. Let T_{ss} be the matrix obtained by converting all the off-diagonal 1's to 0's, making T diagonal. Show that T_{ss} is in $k[T]$.

[08.22] Let $M = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$ be a matrix in a block decomposition, where A is m-by-m and D is n-by-n. Show that

$$\det M = \det A \cdot \det D$$

[08.23] The so-called *Kronecker product*[1] of an m-by-m matrix A and an n-by-n matrix B is

$$A \otimes B = \begin{pmatrix} A_{11} \cdot B & A_{12} \cdot B & \ldots & A_{1m} \cdot B \\ A_{21} \cdot B & A_{22} \cdot B & \ldots & A_{2m} \cdot B \\ & \vdots & & \\ A_{m1} \cdot B & A_{m2} \cdot B & \ldots & A_{mm} \cdot B \end{pmatrix}$$

[1] As we will see shortly, this is really a **tensor product**, and we will treat this question more sensibly.

where, as it may appear, the matrix B is inserted as n-by-n blocks, multiplied by the respective entries A_{ij} of A. Prove that

$$\det(A \otimes B) = (\det A)^n \cdot (\det B)^m$$

at least for $m = n = 2$.

[08.24] For distinct primes p, q, compute

$$\mathbb{Z}/p \otimes_{\mathbb{Z}/pq} \mathbb{Z}/q$$

where for a divisor d of an integer n the abelian group $\mathbb{Z}/d$ is given the $\mathbb{Z}/n$-module structure by

$$(r + n\mathbb{Z}) \cdot (x + d\mathbb{Z}) = rx + d\mathbb{Z}$$

[08.25] Compute $\mathbb{Z}/n \otimes_{\mathbb{Z}} \mathbb{Q}$ with $0 < n \in \mathbb{Z}$.

[08.26] Compute $\mathbb{Z}/n \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}$ with $0 < n \in \mathbb{Z}$.

[08.27] Compute $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Q}/\mathbb{Z})$ for $0 < n \in \mathbb{Z}$.

[08.28] Compute $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$.

[08.29] Compute $(\mathbb{Q}/\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$.

[08.30] Compute $(\mathbb{Q}/\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})$.

[08.31] Prove that for a subring R of a commutative ring S, with $1_R = 1_S$, polynomial rings $R[x]$ behave well with respect to tensor products, namely that (as rings)

$$R[x] \otimes_R S \approx S[x]$$

[08.32] Let K be a field extension of a field k. Let $f(x) \in k[x]$. Show that

$$k[x]/f \otimes_k K \approx K[x]/f$$

where the indicated quotients are by the ideals generated by f in $k[x]$ and $K[x]$, respectively.

[08.33] Let K be a field extension of a field k. Let V be a finite-dimensional k-vectorspace. Show that $V \otimes_k K$ is a good definition of the **extension of scalars** of V from k to K, in the sense that for any K-vectorspace W

$$\mathrm{Hom}_K(V \otimes_k K, W) \approx \mathrm{Hom}_k(V, W)$$

where in $\mathrm{Hom}_k(V, W)$ we *forget* that W was a K-vectorspace, and only think of it as a k-vectorspace.

[08.34] Let M and N be free R-modules, where R is a commutative ring with identity. Prove that $M \otimes_R N$ is free and

$$\mathrm{rank}\, M \otimes_R N = \mathrm{rank}\, M \cdot \mathrm{rank}\, N$$

[08.35] Let M be a free R-module of rank r, where R is a commutative ring with identity. Let S be a commutative ring with identity containing R, such that $1_R = 1_S$. Prove that as an S module $M \otimes_R S$ is free of rank r.

[08.36] For finite-dimensional vectorspaces V, W over a field k, prove that there is a natural isomorphism

$$(V \otimes_k W)^* \approx V^* \otimes W^*$$

where $X^* = \mathrm{Hom}_k(X, k)$ for a k-vectorspace X.

[08.37] For a finite-dimensional k-vectorspace V, prove that the bilinear map

$$B : V \times V^* \to \mathrm{End}_k(V)$$

by

$$B(v \times \lambda)(x) = \lambda(x) \cdot v$$

gives an isomorphism $V \otimes_k V^* \to \mathrm{End}_k(V)$. Further, show that the composition of endormorphisms is the same as the map induced from the map on

$$(V \otimes V^*) \times (V \otimes V^*) \to V \otimes V^*$$

given by

$$(v \otimes \lambda) \times (w \otimes \mu) \to \lambda(w) v \otimes \mu$$

[08.38] Under the isomorphism of the previous problem, show that the linear map

$$\mathrm{tr} : \mathrm{End}_k(V) \to k$$

is the linear map

$$V \otimes V^* \to k$$

induced by the bilinear map $v \times \lambda \to \lambda(v)$.

[08.39] Prove that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ for two endomorphisms of a finite-dimensional vector space V over a field k, with trace defined as just above.

[08.40] Prove the *expansion by minors* formula for determinants, namely, for an n-by-n matrix A with entries a_{ij}, letting A^{ij} be the matrix obtained by deleting the i^{th} row and j^{th} column, for any fixed row index i,

$$\det A = (-1)^i \sum_{j=1}^{n} (-1)^j \, a_{ij} \, \det A^{ij}$$

and symmetrically for expansion along a column.

[08.41] Let M and N be free R-modules, where R is a commutative ring with identity. Prove that $M \otimes_R N$ is free and

$$\mathrm{rank}\, M \otimes_R N = \mathrm{rank}\, M \cdot \mathrm{rank}\, N$$

[08.42] Let M be a free R-module of rank r, where R is a commutative ring with identity. Let S be a commutative ring with identity containing R, such that $1_R = 1_S$. Prove that as an S module $M \otimes_R S$ is free of rank r.

[08.43] For finite-dimensional vectorspaces V, W over a field k, prove that there is a natural isomorphism

$$(V \otimes_k W)^* \approx V^* \otimes W^*$$

where $X^* = \mathrm{Hom}_k(X, k)$ for a k-vectorspace X.

[08.44] For a finite-dimensional k-vectorspace V, prove that the bilinear map

$$B : V \times V^* \to \mathrm{End}_k(V)$$

by

$$B(v \times \lambda)(x) = \lambda(x) \cdot v$$

gives an isomorphism $V \otimes_k V^* \to \mathrm{End}_k(V)$. Further, show that the composition of endormorphisms is the same as the map induced from the map on

$$(V \otimes V^*) \times (V \otimes V^*) \to V \otimes V^*$$

given by

$$(v \otimes \lambda) \times (w \otimes \mu) \to \lambda(w) v \otimes \mu$$

[08.45] Via the isomorphism $\mathrm{End}_k(V) \approx V \otimes_k V^*$, show that the linear map

$$\mathrm{tr} : \mathrm{End}_k(V) \to k$$

is the linear map

$$V \otimes V^* \to k$$

induced by the bilinear map $v \times \lambda \to \lambda(v)$.

[08.46] Prove that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ for two endomorphisms of a finite-dimensional vector space V over a field k, with trace defined as just above.

[08.47] Prove that tensor products are *associative*, in the sense that, for R-modules A, B, C, we have a *natural isomorphism*

$$A \otimes_R (B \otimes_R C) \approx (A \otimes_R B) \otimes_R C$$

In particular, *do* prove the *naturality*, at least the one-third part of it which asserts that, for every R-module homomorphism $f : A \to A'$, the diagram

$$\begin{array}{ccc} A \otimes_R (B \otimes_R C) & \xrightarrow{\approx} & (A \otimes_R B) \otimes_R C \\ \downarrow{\scriptstyle f\otimes(1_B \otimes 1_C)} & & \downarrow{\scriptstyle (f\otimes 1_B)\otimes 1_C} \\ A' \otimes_R (B \otimes_R C) & \xrightarrow{\approx} & (A' \otimes_R B) \otimes_R C \end{array}$$

commutes, where the two horizontal isomorphisms are those determined in the first part of the problem. (One might also consider maps $g : B \to B'$ and $h : C \to C'$, but these behave similarly, so there's no real compulsion to worry about them, apart from awareness of the issue.)

[08.48] Consider the injection $\mathbb{Z}/2 \xrightarrow{t} \mathbb{Z}/4$ which maps

$$t : x + 2\mathbb{Z} \to 2x + 4\mathbb{Z}$$

Show that the induced map

$$t \otimes 1_{\mathbb{Z}/2} : \mathbb{Z}/2 \otimes_{\mathbb{Z}} \mathbb{Z}/2 \to \mathbb{Z}/4 \otimes_{\mathbb{Z}} \mathbb{Z}/2$$

is no longer an injection.

[08.49] Prove that if $s : M \to N$ is a *surjection* of $\mathbb{Z}$-modules and X is any other $\mathbb{Z}$ module, then the induced map

$$s \otimes 1_Z : M \otimes_{\mathbb{Z}} X \to N \otimes_{\mathbb{Z}} X$$

is still surjective.

[08.50] Give an example of a surjection $f : M \to N$ of $\mathbb{Z}$-modules, and another $\mathbb{Z}$-module X, such that the induced map

$$f \circ - : \mathrm{Hom}_{\mathbb{Z}}(X, M) \to \mathrm{Hom}_{\mathbb{Z}}(X, N)$$

(by post-composing) *fails* to be surjective.

[08.51] Let $G : \{\mathbb{Z} - \text{modules}\} \to \{\text{sets}\}$ be the functor that forgets that a module is a module, and just retains the underlying set. Let $F : \{\text{sets}\} \to \{\mathbb{Z} - \text{modules}\}$ be the functor which creates the free module FS on the set S (*and* keeps in mind a map $i : S \to FS$). Show that for any set S and any $\mathbb{Z}$-module M

$$\mathrm{Hom}_{\mathbb{Z}}(FS, M) \approx \mathrm{Hom}_{\text{sets}}(S, GM)$$

Prove that the isomorphism you describe is *natural* in S. (It is also natural in M, but don't prove this.)

[08.52] Let $M = \begin{pmatrix} m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}$ be a 2-by-3 integer matrix, such that the *gcd* of the three 2-by-2 minors is 1. Prove that there exist three integers m_{11}, m_{12}, m_{33} such that

$$\det \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix} = 1$$

[08.53] Let a, b, c be integers whose *gcd* is 1. Prove (without manipulating matrices) that there is a 3-by-3 integer matrix with top row $(a\ b\ c)$ with determinant 1.

[08.54] Let

$$M = \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} & m_{15} \\ m_{21} & m_{22} & m_{23} & m_{24} & m_{25} \\ m_{31} & m_{32} & m_{33} & m_{34} & m_{35} \end{pmatrix}$$

and suppose that the *gcd* of all determinants of 3-by-3 minors is 1. Prove that there exists a 5-by-5 integer matrix $\tilde{M}$ with M as its top 3 rows, such that $\det \tilde{M} = 1$.

[08.55] Let R be a commutative ring with unit. For a *finitely-generated* free R-module F, prove that there is a (natural) isomorphism

$$\mathrm{Hom}_R(F, R) \approx F$$

Or is it only

$$\mathrm{Hom}_R(R, F) \approx F$$

instead? (*Hint:* Recall the definition of a free module.)

[08.56] Let R be an integral domain. Let M and N be free R-modules of finite ranks r, s, respectively. Suppose that there is an R-bilinear map

$$B : M \times N \to R$$

which is *non-degenerate* in the sense that for every $0 \neq m \in M$ there is $n \in N$ such that $B(m, n) \neq 0$, and vice-versa. Prove that $r = s$.

[08.57] Let $\varphi : R \to S$ be commutative rings with unit, and suppose that $\varphi(1_R) = 1_S$, thus making S an R-algebra. For an R-module N prove that $\mathrm{Hom}_R(S, N)$ is (*yet another*) good definition of *extension of scalars* from R to S, by checking that for every S-module M there is a natural isomorphism

$$\mathrm{Hom}_R(\mathrm{Res}^S_R M, N) \approx \mathrm{Hom}_S(M, \mathrm{Hom}_R(S, N))$$

where $\mathrm{Res}^S_R M$ is the R-module obtained by forgetting S, and letting $r \in R$ act on M by $r \cdot m = \varphi(r)m$. (*Do* prove naturality in M, also.)

[08.58] Let

$$M = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \qquad N = \mathbb{Z} \oplus 4\mathbb{Z} \oplus 24\mathbb{Z} \oplus 144\mathbb{Z}$$

What are the elementary divisors of $\bigwedge^2(M/N)$?
