[18.1] Let k be a field, and V a finite-dimensional k vectorspace. Let Λ be a subset of the dual space V^*, with $|\Lambda| < \dim V$. Show that the **homogeneous system of equations**

$$\lambda(v) = 0 \ \ (\text{for all } \lambda \in \Lambda)$$

has a non-trivial (that is, non-zero) solution $v \in V$ (meeting all these conditions).

The dimension of the span W of Λ is strictly less than $\dim V^*$, which we've proven is $\dim V^* = \dim V$. We may also identify $V \approx V^{**}$ via the natural isomorphism. With that identification, we may say that the set of solutions is $W^\perp$, and

$$\dim(W^\perp) + \dim W = \dim V^* = \dim V$$

Thus, $\dim W^\perp > 0$, so there are non-zero solutions. ///

[18.2] Let k be a field, and V a finite-dimensional k vectorspace. Let Λ be a *linearly independent* subset of the dual space V^*. Let $\lambda \to a_\lambda$ be a set map $\Lambda \to k$. Show that an **inhomogeneous system of equations**

$$\lambda(v) = a_\lambda \ \ (\text{for all } \lambda \in \Lambda)$$

has a solution $v \in V$ (meeting all these conditions).

Let $m = |\Lambda|$, $\Lambda = \{\lambda_1, \ldots, \lambda_m\}$. One way to use the linear independence of the functionals in Λ is to extend Λ to a basis $\lambda_1, \ldots, \lambda_n$ for V^*, and let $e_1, \ldots, e_n \in V^{**}$ be the corresponding dual basis for V^{**}. Then let $v_1, \ldots, v_n$ be the images of the e_i in V under the natural isomorphism $V^{**} \approx V$. (This achieves the effect of making the λ_i be a dual basis to the v_i. We had only literally proven that one can go from a basis of a vector space to a dual basis of its dual, and not the reverse.) Then

$$v = \sum_{1 \le i \le m} a_{\lambda_i} \cdot v_i$$

is a solution to the indicated set of equations, since

$$\lambda_j(v) = \sum_{1 \le i \le m} a_{\lambda_i} \cdot \lambda_j(v_i) = a_{\lambda_j}$$

for all indices $j \le m$. ///

[18.3] Let T be a k-linear endomorphism of a finite-dimensional k-vectorspace V. For an eigenvalue λ of T, let V_λ be the generalized λ-eigenspace

$$V_\lambda = \{v \in V : (T - \lambda)^n v = 0 \text{ for some } 1 \le n \in \mathbb{Z}\}$$

Show that the projector P of V to V_λ (commuting with T) lies inside $k[T]$.

First we do this assuming that the minimal polynomial of T factors into linear factors in $k[x]$.

Let $f(x)$ be the minimal polynomial of T, and let $f_\lambda(x) = f(x)/(x-\lambda)^e$ where $(x-\lambda)^e$ is the precise power of $(x - \lambda)$ dividing $f(x)$. Then the collection of all $f_\lambda(x)$'s has *gcd* 1, so there are $a_\lambda(x) \in k[x]$ such that

$$1 = \sum_\lambda a_\lambda(x)\, f_\lambda(x)$$

We claim that $E_\lambda = a_\lambda(T) f_\lambda(T)$ is a projector to the generalized λ-eigenspace V_λ. Indeed, for $v \in V_\lambda$,

$$v = 1_V \cdot v = \sum_\mu a_\mu(T) f_\mu(T) \cdot v = \sum_\mu a_\mu(T) f_\mu(T) \cdot v = a_\lambda(T) f_\lambda(T) \cdot v$$

since $(x-\lambda)^e$ divides $f_\mu(x)$ for $\mu \neq \lambda$, and $(T-\lambda)^e v = 0$. That is, it acts as the identity on V_λ. And

$$(T-\lambda)^e \circ E_\lambda = a_\lambda(T)\, f(T) = 0 \in \mathrm{End}_k(V)$$

so the image of E_λ is inside V_λ. Since E_λ is the identity on V_λ, it must be that the image of E_λ is *exactly* V_λ. For $\mu \neq \lambda$, since $f(x) | f_\mu(x) f_\lambda(x)$, $E_\mu E_\lambda = 0$, so these idempotents are *mutually orthogonal.* Then

$$(a_\lambda(T) f_\lambda(T))^2 = (a_\lambda(T) f_\lambda(T)) \cdot (1 - \sum_{\mu \neq \lambda} a_\mu(T) f_\mu(T)) = a_\lambda(T) f_\lambda(T) - 0$$

That is, $E_\lambda^2 = E_\lambda$, so E_λ is *a* projector to V_λ.

The mutual orthogonality of the idempotents will yield the fact that V is the direct sum of all the generalized eigenspaces of T. Indeed, for any $v \in V$,

$$v = 1 \cdot v = (\sum_\lambda E_\lambda)\, v = \sum_\lambda (E_\lambda v)$$

and $E_\lambda v \in V_\lambda$. Thus,

$$\sum_\lambda V_\lambda = V$$

To check that the sum is (unsurprisingly) direct, let $v_\lambda \in V_\lambda$, and suppose

$$\sum_\lambda v_\lambda = 0$$

Then $v_\lambda = E_\lambda v_\lambda$, for all λ. Then apply E_μ and invoke the orthogonality of the idempotents to obtain

$$v_\mu = 0$$

This proves the linear independence, and that the sum is direct.

To prove *uniqueness* of a projector E to V_λ commuting with T, note that any operator S commuting with T necessarily stabilizes all the generalized eigenspaces of T, since for $v \in V_\mu$

$$(T-\lambda)^e\, Sv = S\,(T-\lambda)^e v = S \cdot 0 = 0$$

Thus, E stabilizes all the V_μs. Since V is the direct sum of the V_μ and E maps V to V_λ, it must be that E is 0 on V_μ for $\mu \neq \lambda$. Thus,

$$E = 1 \cdot E_\lambda + \sum_{\mu \neq \lambda} 0 \cdot E_\mu = E_\lambda$$

That is, there is just one projector to V_λ that also commutes with T. *This finishes things under the assumption that $f(x)$ factors into linear factors in $k[x]$.*

The more general situation is similar. More generally, for a monic irreducible $P(x)$ in $k[x]$ dividing $f(x)$, with $P(x)^e$ the precise power of $P(x)$ dividing $f(x)$, let

$$f_P(x) = f(x)/P(x)^e$$

Then these f_P have *gcd* 1, so there are $a_P(x)$ in $k[x]$ such that

$$1 = \sum_P a_P(x) \cdot f_P(x)$$

Let $E_P = a_P(T) f_P(T)$. Since $f(x)$ divides $f_P(x) \cdot f_Q(x)$ for distinct irreducibles P, Q, we have $E_P \circ E_Q = 0$ for $P \neq Q$. And

$$E_P^2 = E_P(1 - \sum_{Q \neq P} E_Q) = E_P$$

so (as in the simpler version) the E_P's are mutually orthogonal idempotents. And, similarly, V is the direct sum of the subspaces

$$V_P = E_P \cdot V$$

We can also characterize V_P as the kernel of $P^e(T)$ on V, where $P^e(x)$ is the power of $P(x)$ dividing $f(x)$. If $P(x) = (x - \lambda)$, then V_P is the generalized λ-eigenspace, and E_P is the projector to it.

If E were another projector to V_λ commuting with T, then E stabilizes V_P for all irreducibles P dividing the minimal polynomial f of T, and E is 0 on V_Q for $Q \neq (x - \lambda)$, and E is 1 on V_λ. That is,

$$E = 1 \cdot E_{x-\lambda} + \sum_{Q \neq x-\lambda} 0 \cdot E_Q = E_P$$

This proves the uniqueness even in general. ///

[18.4] Let T be a matrix in Jordan normal form with entries in a field k. Let T_{ss} be the matrix obtained by converting all the off-diagonal 1's to 0's, making T diagonal. Show that T_{ss} is in $k[T]$.

This implicitly demands that the minimal polynomial of T factors into linear factors in $k[x]$.

Continuing as in the previous example, let $E_\lambda \in k[T]$ be the projector to the generalized λ-eigenspace V_λ, and keep in mind that we have shown that V is the direct sum of the generalized eigenspaces, equivalent, that $\sum_\lambda E_\lambda = 1$. By definition, the operator T_{ss} is the scalar operator λ on V_λ. Then

$$T_{ss} = \sum_\lambda \lambda \cdot E_\lambda \in k[T]$$

since (from the previous example) each E_λ is in $k[T]$. ///

[18.5] Let $M = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$ be a matrix in a block decomposition, where A is m-by-m and D is n-by-n. Show that

$$\det M = \det A \cdot \det D$$

One way to prove this is to use the formula for the determinant of an N-by-N matrix

$$\det C = \sum_{\pi \in S_N} \sigma(\pi)\, a_{\pi(1),1} \ldots a_{\pi(N),N}$$

where c_{ij} is the $(i,j)^{th}$ entry of C, π is summed over the symmetric group S_N, and σ is the sign homomorphism. Applying this to the matrix M,

$$\det M = \sum_{\pi \in S_{m+n}} \sigma(\pi)\, M_{\pi(1),1} \ldots M_{\pi(m+n),m+n}$$

where M_{ij} is the $(i,j)^{th}$ entry. Since the entries M_{ij} with $1 \leq j \leq m$ and $m < i \leq m+n$ are all 0, we should only sum over π with the property that

$$\pi(j) \leq m \quad \text{for} \quad 1 \leq j \leq m$$

That is, π stabilizes the subset $\{1,\ldots,m\}$ of the indexing set. Since π is a bijection of the index set, necessarily such π stabilizes $\{m+1,m+2,\ldots,m+n\}$, also. Conversely, each pair (π_1,π_2) of permutation π_1 of the first m indices and π_2 of the last n indices gives a permutation of the whole set of indices.

Let X be the set of the permutations $\pi \in S_{m+n}$ that stabilize $\{1,\ldots,m\}$. For each $\pi \in X$, let π_1 be the restriction of π to $\{1,\ldots,m\}$, and let π_2 be the restriction to $\{m+1,\ldots,m+n\}$. And, in fact, if we plan to index the entries of the block D in the usual way, we'd better be able to think of π_2 as a permutation of $\{1,\ldots,n\}$, also. Note that $\sigma(\pi)=\sigma(\pi_1)\sigma(\pi_2)$. Then

$$\det M = \sum_{\pi\in X} \sigma(\pi)\, M_{\pi(1),1}\ldots M_{\pi(m+n),m+n}$$

$$= \sum_{\pi\in X} \sigma(\pi)\, (M_{\pi(1),1}\ldots M_{\pi(m),m})\cdot (M_{\pi(m+1),m+1}\ldots M_{\pi(m+n),m+n})$$

$$= \left(\sum_{\pi_1\in S_m} \sigma(\pi_1)\, M_{\pi_1(1),1}\ldots M_{\pi_1(m),m}\right)\cdot\left(\sum_{\pi_2\in S_n} \sigma(\pi_2)(M_{\pi_2(m+1),m+1}\ldots M_{\pi_2(m+n),m+n}\right)$$

$$= \left(\sum_{\pi_1\in S_m} \sigma(\pi_1)\, A_{\pi_1(1),1}\ldots A_{\pi_1(m),m}\right)\cdot\left(\sum_{\pi_2\in S_n} \sigma(\pi_2)D_{\pi_2(1),1}\ldots D_{\pi_2(n),n}\right) = \det A\cdot\det D$$

where in the last part we have mapped $\{m+1,\ldots,m+n\}$ bijectively by $\ell \to \ell - m$. ///

[18.6] The so-called *Kronecker product*[1] of an m-by-m matrix A and an n-by-n matrix B is

$$A\otimes B = \begin{pmatrix} A_{11}\cdot B & A_{12}\cdot B & \ldots & A_{1m}\cdot B \\ A_{21}\cdot B & A_{22}\cdot B & \ldots & A_{2m}\cdot B \\ & \vdots & & \\ A_{m1}\cdot B & A_{m2}\cdot B & \ldots & A_{mm}\cdot B \end{pmatrix}$$

where, as it may appear, the matrix B is inserted as n-by-n blocks, multiplied by the respective entries A_{ij} of A. Prove that

$$\det(A\otimes B) = (\det A)^n\cdot(\det B)^m$$

at least for $m=n=2$.

If no entry of the first row of A is non-zero, then both sides of the desired equality are 0, and we're done. So suppose some entry A_{1i} of the first row of A is non-zero. If $i\neq 1$, then for $\ell = 1,\ldots,n$ interchange the ℓ^{th} and $(i-1)n+\ell^{th}$ columns of $A\otimes B$, thus multiplying the determinant by $(-1)^n$. This is compatible with the formula, so we'll assume that $A_{11}\neq 0$ to do an induction on m.

We will manipulate n-by-n *blocks* of scalar multiples of B rather than actual scalars.

Thus, assuming that $A_{11}\neq 0$, we want to subtract multiples of the left column of n-by-n blocks from the blocks further to the right, to make the top n-by-n blocks all 0 (apart from the leftmost block, $A_{11}B$). In terms of manipulations of columns, for $\ell = 1,\ldots,n$ and $j = 2,3,\ldots,m$ subtract A_{1j}/A_{11} times the ℓ^{th} column of $A\otimes B$ from the $((j-1)n+\ell)^{th}$. Since for $1\le \ell\le n$ the ℓ^{th} column of $A\otimes B$ is A_{11} times the ℓ^{th} column of B, and the $((j-1)n+\ell)^{th}$ column of $A\otimes B$ is A_{1j} times the ℓ^{th} column of B, this has the desired effect of killing off the n-by-n blocks along the top of $A\otimes B$ except for the leftmost block. And the $(i,j)^{th}$ n-by-n block of $A\otimes B$ has become $(A_{ij}-A_{1j}A_{i1}/A_{11})\cdot B$. Let

$$A'_{ij} = A_{ij} - A_{1j}A_{i1}/A_{11}$$

[1] As we will see shortly, this is really a **tensor product**, and we will treat this question more sensibly.

and let D be the $(m-1)$-by-$(m-1)$ matrix with $(i,j)^{th}$ entry $D_{ij} = A'_{(i-1),(j-1)}$. Thus, the manipulation so far gives

$$\det(A \otimes B) = \det \begin{pmatrix} A_{11}B & 0 \\ * & D \otimes B \end{pmatrix}$$

By the previous example (or its tranpose)

$$\det \begin{pmatrix} A_{11}B & 0 \\ * & D \otimes B \end{pmatrix} = \det(A_{11}B) \cdot \det(D \otimes B) = A_{11}^n \det B \cdot \det(D \otimes B)$$

by the multilinearity of det.

And, at the same time subtracting A_{1j}/A_{11} times the first column of A from the j^{th} column of A for $2 \leq j \leq m$ does not change the determinant, and the new matrix is

$$\begin{pmatrix} A_{11} & 0 \\ * & D \end{pmatrix}$$

Also by the previous example,

$$\det A = \det \begin{pmatrix} A_{11} & 0 \\ * & D \end{pmatrix} = A_1 1 \cdot \det D$$

Thus, putting the two computations together,

$$\det(A \otimes B) = A_{11}^n \det B \cdot \det(D \otimes B) = A_{11}^n \det B \cdot (\det D)^n (\det B)^{m-1}$$
$$= (A_{11} \det D)^n \det B \cdot (\det B)^{m-1} = (\det A)^n (\det B)^m$$

as claimed.

Another approach to this is to observe that, in these terms, $A \otimes B$ is

$$\begin{pmatrix}
A_{11} & 0 & \ldots & 0 & & A_{1m} & 0 & \ldots & 0 \\
0 & A_{11} & & & & 0 & A_{1m} & & \\
\vdots & & \ddots & & \ldots & \vdots & & \ddots & \\
0 & & & A_{11} & & 0 & & & A_{1m} \\
 & \vdots & & & & & \vdots & & \\
A_{m1} & 0 & \ldots & 0 & & A_{mm} & 0 & \ldots & 0 \\
0 & A_{m1} & & & & 0 & A_{mm} & & \\
\vdots & & \ddots & & \ldots & \vdots & & \ddots & \\
0 & & & A_{m1} & & 0 & & & A_{mm}
\end{pmatrix}
\begin{pmatrix} B & 0 & \ldots & 0 \\ 0 & B & & \\ \vdots & & \ddots & \\ 0 & & & B \end{pmatrix}$$

where there are m copies of B on the diagonal. By suitable permutations of rows and columns (with an interchange of rows for each interchange of columns, thus giving no net change of sign), the matrix containing the A_{ij}s becomes

$$\begin{pmatrix} A & 0 & \ldots & 0 \\ 0 & A & & \\ \vdots & & \ddots & \\ 0 & & & A \end{pmatrix}$$

with n copies of A on the diagonal. Thus,

$$\det(A \otimes B) = \det \begin{pmatrix} A & 0 & \ldots & 0 \\ 0 & A & & \\ \vdots & & \ddots & \\ 0 & & & A \end{pmatrix} \cdot \det \begin{pmatrix} B & 0 & \ldots & 0 \\ 0 & B & & \\ \vdots & & \ddots & \\ 0 & & & B \end{pmatrix} = (\det A)^n \cdot (\det B)^m$$

This might be more attractive than the first argument, depending on one's tastes. ///