[19.1] For distinct primes p, q, compute

$$\mathbb{Z}/p \otimes_{\mathbb{Z}/pq} \mathbb{Z}/q$$

where for a divisor d of an integer n the abelian group $\mathbb{Z}/d$ is given the $\mathbb{Z}/n$-module structure by

$$(r + n\mathbb{Z}) \cdot (x + d\mathbb{Z}) = rx + d\mathbb{Z}$$

We claim that this tensor product is 0. To prove this, it suffices to prove that every $m \otimes n$ (the image of $m \times n$ in the tensor product) is 0, since we have shown that these *monomial* tensors always generate the tensor product.

Since p and q are relatively prime, there exist integers a, b such that $1 = ap + bq$. Then for all $m \in \mathbb{Z}/p$ and $n \in \mathbb{Z}/q$,

$$m \otimes n = 1 \cdot (m \otimes n) = (ap + bq)(m \otimes n) = a(pm \otimes n) + b(m \otimes qn) = a \cdot 0 + b \cdot 0 = 0$$

An auxiliary point is to recognize that, indeed, $\mathbb{Z}/p$ and $\mathbb{Z}/q$ really are $\mathbb{Z}/pq$-modules, and that the equation $1 = ap + bq$ still does make sense inside $\mathbb{Z}/pq$. ///

[19.2] Compute $\mathbb{Z}/n \otimes_{\mathbb{Z}} \mathbb{Q}$ with $0 < n \in \mathbb{Z}$.

We claim that the tensor product is 0. It suffices to show that every $m \otimes n$ is 0, since these monomials generate the tensor product. For any $x \in \mathbb{Z}/n$ and $y \in \mathbb{Q}$,

$$x \otimes y = x \otimes (n \cdot \frac{y}{n}) = (nx) \otimes \frac{y}{n} = 0 \otimes \frac{y}{n} = 0$$

as claimed. ///

[19.3] Compute $\mathbb{Z}/n \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}$ with $0 < n \in \mathbb{Z}$.

We claim that the tensor product is 0. It suffices to show that every $m \otimes n$ is 0, since these monomials generate the tensor product. For any $x \in \mathbb{Z}/n$ and $y \in \mathbb{Q}/\mathbb{Z}$,

$$x \otimes y = x \otimes (n \cdot \frac{y}{n}) = (nx) \otimes \frac{y}{n} = 0 \otimes \frac{y}{n} = 0$$

as claimed. ///

[19.4] Compute $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Q}/\mathbb{Z})$ for $0 < n \in \mathbb{Z}$.

Let $q : \mathbb{Z} \to \mathbb{Z}/n$ be the natural quotient map. Given $\varphi \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Q}/\mathbb{Z})$, the composite $\varphi \circ q$ is a $\mathbb{Z}$-homomorphism from the free $\mathbb{Z}$-module $\mathbb{Z}$ (on one generator 1) to $\mathbb{Q}/\mathbb{Z}$. A homomorphism $\Phi \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$ is completely determined by the image of 1 (since $\Phi(\ell) = \Phi(\ell \cdot 1) = \ell \cdot \Phi(1)$), and since $\mathbb{Z}$ is *free* this image can be *anything* in the target $\mathbb{Q}/\mathbb{Z}$.

Such a homomorphism $\Phi \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$ factors through $\mathbb{Z}/n$ if and only if $\Phi(n) = 0$, that is, $n \cdot \Phi(1) = 0$. A complete list of representatives for equivalence classes in $\mathbb{Q}/\mathbb{Z}$ annihilated by n is $0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \ldots, \frac{n-1}{n}$. Thus, $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Q}/\mathbb{Z})$ is in bijection with this set, by

$$\varphi_{i/n}(x + n\mathbb{Z}) = ix/n + \mathbb{Z}$$

In fact, we see that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Q}/\mathbb{Z})$ is an abelian group isomorphic to $\mathbb{Z}/n$, with

$$\varphi_{1/n}(x + n\mathbb{Z}) = x/n + \mathbb{Z}$$

as a generator. ///

[19.5] Compute $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$.

We claim that this tensor product is isomorphic to $\mathbb{Q}$, via the $\mathbb{Z}$-linear map β induced from the $\mathbb{Z}$-bilinar map $B : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$ given by

$$B : x \times y \to xy$$

First, observe that the monomials $x \otimes 1$ generate the tensor product. Indeed, given $a/b \in \mathbb{Q}$ (with a, b integers, $b \neq 0$) we have

$$x \otimes \frac{a}{b} = (\frac{x}{b} \cdot b) \otimes \frac{a}{b} = \frac{x}{b} \otimes (b \cdot \frac{a}{b}) = \frac{x}{b} \otimes a = \frac{x}{b} \otimes a \cdot 1 = (a \cdot \frac{x}{b}) \otimes 1 = \frac{ax}{b} \otimes 1$$

proving the claim. Further, any finite $\mathbb{Z}$-linear combination of such elements can be rewritten as a single one: letting $n_i \in \mathbb{Z}$ and $x_i \in \mathbb{Q}$, we have

$$\sum_i n_i \cdot (x_i \otimes 1) = (\sum_i n_i x_i) \otimes 1$$

This gives an outer bound for the size of the tensor product. Now we need an inner bound, to know that there is no *further* collapsing in the tensor product.

From the defining property of the tensor product there *exists* a (unique) $\mathbb{Z}$-linear map from the tensor product to $\mathbb{Q}$, through which B factors. We have $B(x, 1) = x$, so the induced $\mathbb{Z}$-linear map β is a bijection on $\{x \otimes 1 : x \in \mathbb{Q}\}$, so it is an isomorphism. ///

[19.6] Compute $(\mathbb{Q}/\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$.

We claim that the tensor product is 0. It suffices to show that every $m \otimes n$ is 0, since these monomials generate the tensor product. Given $x \in \mathbb{Q}/\mathbb{Z}$, let $0 < n \in \mathbb{Z}$ such that $nx = 0$. For any $y \in \mathbb{Q}$,

$$x \otimes y = x \otimes (n \cdot \frac{y}{n}) = (nx) \otimes \frac{y}{n} = 0 \otimes \frac{y}{n} = 0$$

as claimed. ///

[19.7] Compute $(\mathbb{Q}/\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})$.

We claim that the tensor product is 0. It suffices to show that every $m \otimes n$ is 0, since these monomials generate the tensor product. Given $x \in \mathbb{Q}/\mathbb{Z}$, let $0 < n \in \mathbb{Z}$ such that $nx = 0$. For any $y \in \mathbb{Q}/\mathbb{Z}$,

$$x \otimes y = x \otimes (n \cdot \frac{y}{n}) = (nx) \otimes \frac{y}{n} = 0 \otimes \frac{y}{n} = 0$$

as claimed. Note that we do *not* claim that $\mathbb{Q}/Z$ is a $\mathbb{Q}$-module (which it is not), but only that for given $y \in \mathbb{Q}/\mathbb{Z}$ there is another element $z \in \mathbb{Q}/\mathbb{Z}$ such that $nz = y$. That is, $\mathbb{Q}/Z$ is a **divisible** $\mathbb{Z}$-module. ///

[19.8] Prove that for a subring R of a commutative ring S, with $1_R = 1_S$, polynomial rings $R[x]$ behave well with respect to tensor products, namely that (as rings)

$$R[x] \otimes_R S \approx S[x]$$

Given an R-algebra homomorphism $\varphi : R \to A$ and $a \in A$, let $\Phi : R[x] \to A$ be the unique R-algebra homomorphism $R[x] \to A$ which is φ on R and such that $\varphi(x) = a$. In particular, this works for A an

S-algebra and φ the restriction to R of an S-algebra homomorphism $\varphi : S \to A$. By the defining property of the tensor product, the bilinear map $B : R[x] \times S \to A$ given by

$$B(P(x) \times s) = s \cdot \Phi(P(x))$$

gives a unique R-module map $\beta : R[x] \otimes_R S \to A$. Thus, the tensor product has most of the properties necessary for it to be the free S-algebra on one generator $x \otimes 1$.

[0.0.1] **Remark:** However, we might be concerned about verification that each such β is an S-algebra map, rather than just an R-module map. We can certainly write an expression that appears to describe the multiplication, by

$$(P(x) \otimes s) \cdot (Q(x) \otimes t) = P(x)Q(x) \otimes st$$

for polynomials P, Q and $s, t \in S$. *If* it is *well-defined*, then it is visibly associative, distributive, etc., as required.

[0.0.2] **Remark:** The S-module structure itself is more straightforward: for any R-module M the tensor product $M \otimes_R S$ has a natural S-module structure given by

$$s \cdot (m \otimes t) = m \otimes st$$

for $s, t \in S$ and $m \in M$. But one could object that this structure is chosen at random. To argue that this is a *good* way to convert M into an S-module, we claim that for any other S-module N we have a natural isomorphism of abelian groups

$$\mathrm{Hom}_S(M \otimes_R S, N) \approx \mathrm{Hom}_R(M, N)$$

(where on the right-hand side we simply *forget* that N had more structure than that of R-module). The map is given by

$$\Phi \to \varphi_\Phi \quad \text{where} \quad \varphi_\Phi(m) = \Phi(m \otimes 1)$$

and has inverse

$$\Phi_\varphi \longleftarrow \varphi \quad \text{where} \quad \Phi_\varphi(m \otimes s) = s \cdot \varphi(m)$$

One might further carefully verify that these two maps are inverses.

[0.0.3] **Remark:** The definition of the tensor product does give an $\mathbb{R}$-linear map

$$\beta : R[x] \otimes_R S \to S[x]$$

associated to the R-bilinear $B : R[x] \times S \to S[x]$ by

$$B(P(x) \otimes s) = s \cdot P(x)$$

for $P(x) \in R[x]$ and $s \in S$. But it does not seem trivial to prove that this gives an isomorphism. Instead, it may be better to use the universal mapping property of a free algebra. In any case, there would still remain the issue of proving that the induced maps are S-algebra maps.

[19.9] Let K be a field extension of a field k. Let $f(x) \in k[x]$. Show that

$$k[x]/f \otimes_k K \approx K[x]/f$$

where the indicated quotients are by the ideals generated by f in $k[x]$ and $K[x]$, respectively.

Upon reflection, one should realize that we want to prove isomorphism as $K[x]$-modules. Thus, we implicitly use the facts that $k[x]/f$ is a $k[x]$-module, that $k[x] \otimes_k K \approx K[x]$ as K-algebras, and that $M \otimes_k K$ gives a $k[x]$-module M a $K[x]$-module structure by

$$(\sum_i s_i x^i) \cdot (m \otimes 1) = \sum_i (x^i \cdot m) \otimes s_i$$

The map

$$k[x] \otimes_k K \approx_{\text{ring}} K[x] \to K[x]/f$$

has kernel (in $K[x]$) exactly of multiples $Q(x) \cdot f(x)$ of $f(x)$ by polynomials $Q(x) = \sum_i s_i x^i$ in $K[x]$. The inverse image of such a polynomial via the isomorphism is

$$\sum_i x^i f(x) \otimes s_i$$

Let I be the ideal generated in $k[x]$ by f, and $\tilde{I}$ the ideal generated by f in $K[x]$. The k-bilinear map

$$k[x]/f \times K \to K[x]/f$$

by

$$B : (P(x) + I) \times s \to s \cdot P(x) + \tilde{I}$$

gives *a* map $\beta : k[x]/f \otimes_k K \to K[x]/f$. The map β is *surjective*, since

$$\beta(\sum_i (x^i + I) \otimes s_i) = \sum_i s_i x^i + \tilde{I}$$

hits every polynomial $\sum_i s_i x^i$ mod $\tilde{I}$. On the other hand, if

$$\beta(\sum_i (x^i + I) \otimes s_i) \in \tilde{I}$$

then $\sum_i s_i x^i = F(x) \cdot f(x)$ for some $F(x) \in K[x]$. Let $F(x) = \sum_j t_j x^j$. With $f(x) = \sum_\ell c_\ell x^\ell$, we have

$$s_i = \sum_{j+\ell=i} t_j c_\ell$$

Then, using k-linearity,

$$\sum_i (x^i + I) \otimes s_i = \sum_i \left(x^i + I \otimes (\sum_{j+\ell=i} t_j c_\ell) \right) = \sum_{j,\ell} \left(x^{j+\ell} + I \otimes t_j c_\ell \right)$$

$$= \sum_{j,\ell} \left(c_\ell x^{j+\ell} + I \otimes t_j \right) = \sum_j (\sum_\ell c_\ell x^{j+\ell} + I) \otimes t_j = \sum_j (f(x) x^j + I) \otimes t_j = \sum_j 0 = 0$$

So the map is a bijection, so is an isomorphism. ///

[19.10] Let K be a field extension of a field k. Let V be a finite-dimensional k-vectorspace. Show that $V \otimes_k K$ is a good definition of the **extension of scalars** of V from k to K, in the sense that for any K-vectorspace W

$$\mathrm{Hom}_K(V \otimes_k K, W) \approx \mathrm{Hom}_k(V, W)$$

where in $\mathrm{Hom}_k(V, W)$ we *forget* that W was a K-vectorspace, and only think of it as a k-vectorspace.

This is a special case of a general phenomenon regarding *extension of scalars*. For any k-vectorspace V the tensor product $V \otimes_k K$ has a natural K-module structure given by

$$s \cdot (v \otimes t) = v \otimes st$$

for $s, t \in K$ and $v \in V$. To argue that this is a *good* way to convert k-vectorspaces V into K-vectorspaces, claim that for any other K-module W have a natural isomorphism of abelian groups

$$\mathrm{Hom}_K(V \otimes_k K, W) \approx \mathrm{Hom}_k(V, W)$$

On the right-hand side we *forget* that W had more structure than that of k-vectorspace. The map is

$$\Phi \to \varphi_\Phi \quad \text{where} \quad \varphi_\Phi(v) = \Phi(v \otimes 1)$$

and has inverse

$$\Phi_\varphi \longleftarrow \varphi \quad \text{where} \quad \Phi_\varphi(v \otimes s) = s \cdot \varphi(v)$$

To verify that these are mutual inverses, compute

$$\varphi_{\Phi_\varphi}(v) = \Phi_\varphi(v \otimes 1) = 1 \cdot \varphi(v) = \varphi(v)$$

and

$$\Phi_{\varphi_\Phi}(v \otimes 1) = 1 \cdot \varphi_\Phi(v) = \Phi(v \otimes 1)$$

which proves that the maps are inverses. ///

[0.0.4] **Remark:** In fact, the two spaces of homomorphisms in the isomorphism can be given natural structures of K-vectorspaces, and the isomorphism just constructed can be verified to respect this additional structure. The K-vectorspace structure on the left is clear, namely

$$(s \cdot \Phi)(m \otimes t) = \Phi(m \otimes st) = s \cdot \Phi(m \otimes t)$$

The structure on the right is

$$(s \cdot \varphi)(m) = s \cdot \varphi(m)$$

The latter has only the one presentation, since only W is a K-vectorspace.

[19.11] Let M and N be free R-modules, where R is a commutative ring with identity. Prove that $M \otimes_R N$ is free and

$$\mathrm{rank}\, M \otimes_R N = \mathrm{rank}\, M \cdot \mathrm{rank}\, N$$

Let M and N be free on generators $i : X \to M$ and $j : Y \to N$. We claim that $M \otimes_R N$ is free on a set map

$$\ell : X \times Y \to M \otimes_R N$$

To verify this, let $\varphi : X \times Y \to Z$ be a set map. For each fixed $y \in Y$, the map $x \to \varphi(x, y)$ factors through a unique R-module map $B_y : M \to Z$. For each $m \in M$, the map $y \to B_y(m)$ gives rise to a unique R-linear map $n \to B(m, n)$ such that

$$B(m, j(y)) = B_y(m)$$

The linearity in the second argument assures that we still have the linearity in the first, since for $n = \sum_t r_t\, j(y_t)$ we have

$$B(m, n) = B(m, \sum_t r_t j(y_t)) = \sum_t r_t B_{y_t}(m)$$

which is a linear combination of linear functions. Thus, there is a unique map to Z induced on the tensor product, showing that the tensor product with set map $i \times j : X \times Y \to M \otimes_R N$ is free. ///

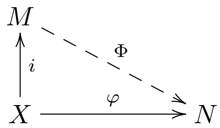
[19.12] Let M be a free R-module of rank r, where R is a commutative ring with identity. Let S be a commutative ring with identity containing R, such that $1_R = 1_S$. Prove that as an S module $M \otimes_R S$ is free of rank r.

We prove a bit more. First, instead of simply an *inclusion* $R \subset S$, we can consider any ring homomorphism $\psi : R \to S$ such that $\psi(1_R) = 1_S$.

Also, we can consider arbitrary sets of generators, and give more details. Let M be free on generators $i : X \to M$, where X is a set. Let $\tau : M \times S \to M \otimes_R S$ be the canonical map. We claim that $M \otimes_R S$ is free on $j : X \to M \otimes_R S$ defined by

$$j(x) = \tau(i(x) \times 1_S)$$

Given an S-module N, we can be a little forgetful and consider N as an R-module via ψ, by $r \cdot n = \psi(r)n$. Then, given a set map $\varphi : X \to N$, since M is free, there is a unique R-module map $\Phi : M \to N$ such that $\varphi = \Phi \circ i$. That is, the diagram



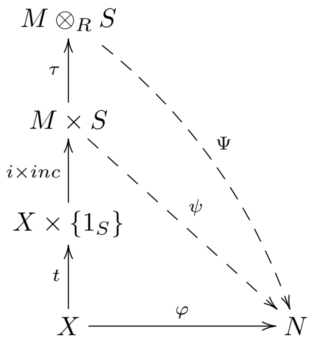
commutes. Then the map

$$\psi : M \times S \to N$$

by

$$\psi(m \times s) = s \cdot \Phi(m)$$

induces (by the defining property of $M \otimes_R S$) a unique $\Psi : M \otimes_R S \to N$ making a commutative diagram



where *inc* is the inclusion map $\{1_S\} \to S$, and where $t : X \to X \times \{1_S\}$ by $x \to x \times 1_S$. Thus, $M \otimes_R S$ is free on the composite $j : X \to M \otimes_R S$ defined to be the composite of the vertical maps in that last diagram. This argument does not depend upon finiteness of the generating set. ///

[19.13] For finite-dimensional vectorspaces V, W over a field k, prove that there is a natural isomorphism

$$(V \otimes_k W)^* \approx V^* \otimes W^*$$

where $X^* = \mathrm{Hom}_k(X, k)$ for a k-vectorspace X.

For finite-dimensional V and W, since $V \otimes_k W$ is free on the cartesian product of the generators for V and W, the dimensions of the two sides match. We make an isomorphism from right to left. Create a bilinear map

$$V^* \times W^* \to (V \otimes_k W)^*$$

as follows. Given $\lambda \in V^*$ and $\mu \in W^*$, as usual make $\Lambda_{\lambda,\mu} \in (V \otimes_k W)^*$ from the bilinear map

$$B_{\lambda,\mu} : V \times W \to k$$

defined by

$$B_{\lambda,\mu}(v, w) = \lambda(v) \cdot \mu(w)$$

This induces a unique functional $\Lambda_{\lambda,\mu}$ on the tensor product. This induces a unique linear map

$$V^* \otimes W^* \to (V \otimes_k W)^*$$

as desired.

Since everything is finite-dimensional, bijectivity will follow from injectivity. Let $e_1, \ldots, e_m$ be a basis for V, $f_1, \ldots, f_n$ a basis for W, and $\lambda_1, \ldots, \lambda_m$ and $\mu_1, \ldots, \mu_n$ corresponding dual bases. We have shown that a basis of a tensor product of free modules is free on the cartesian product of the generators. Suppose that $\sum_{ij} c_{ij} \lambda_i \otimes \mu_j$ gives the 0 functional on $V \otimes W$, for some scalars c_{ij}. Then, for every pair of indices s, t, the function is 0 on $e_s \otimes f_t$. That is,

$$0 = \sum_{ij} c_{ij} \lambda_i(e_s)\, \lambda_j(f_t) = c_{st}$$

Thus, all constants c_{ij} are 0, proving that the map is injective. Then a dimension count proves the isomorphism. ///

[19.14] For a finite-dimensional k-vectorspace V, prove that the bilinear map

$$B : V \times V^* \to \mathrm{End}_k(V)$$

by

$$B(v \times \lambda)(x) = \lambda(x) \cdot v$$

gives an isomorphism $V \otimes_k V^* \to \mathrm{End}_k(V)$. Further, show that the composition of endormorphisms is the same as the map induced from the map on

$$(V \otimes V^*) \times (V \otimes V^*) \to V \otimes V^*$$

given by

$$(v \otimes \lambda) \times (w \otimes \mu) \to \lambda(w) v \otimes \mu$$

The bilinear map $v \times \lambda \to T_{v,\lambda}$ given by

$$T_{v,\lambda}(w) = \lambda(w) \cdot v$$

induces a *unique* linear map $j : V \otimes V^* \to \mathrm{End}_k(V)$.

To prove that j is injective, we may use the fact that a basis of a tensor product of free modules is free on the cartesian product of the generators. Thus, let $e_1, \ldots, e_n$ be a basis for V, and $\lambda_1, \ldots, \lambda_n$ a dual basis for V^*. Suppose that

$$\sum_{i,j=1}^{n} c_{ij}\, e_i \otimes \lambda_j \to 0 \mathrm{End}_k(V)$$

That is, for every e_ℓ,

$$\sum_{ij} c_{ij} \lambda_j(e_\ell) e_i = 0 \in V$$

This is

$$\sum_i c_{ij} e_i = 0 \quad \text{(for all } j)$$

Since the e_is are linearly independent, all the c_{ij}s are 0. Thus, the map j is injective. Then counting k-dimensions shows that this j is a k-linear isomorphism.

Composition of endomorphisms is a bilinear map

$$\mathrm{End}_k(V) \times \mathrm{End}_k(V) \xrightarrow{\circ} \mathrm{End}_k(V)$$

by

$$S \times T \to S \circ T$$

Denote by

$$c : (v \otimes \lambda) \times (w \otimes \mu) \to \lambda(w) v \otimes \mu$$

the allegedly corresonding map on the tensor products. The induced map on $(V \otimes V^*) \otimes (V \otimes V^*)$ is an example of a **contraction map** on tensors. We want to show that the diagram

$$\begin{array}{ccc} \mathrm{End}_k(V) \times \mathrm{End}_k(V) & \xrightarrow{\ \circ\ } & \mathrm{End}_k(V) \\ {\scriptstyle j \times j} \uparrow & & \uparrow {\scriptstyle j} \\ (V \otimes_k V^*) \times (V \otimes_k V^*) & \xrightarrow{\ c\ } & V \otimes_k V^* \end{array}$$

commutes. It suffices to check this starting with $(v \otimes \lambda) \times (w \otimes \mu)$ in the lower left corner. Let $x \in V$. Going up, then to the right, we obtain the endomorphism which maps x to

$$j(v \otimes \lambda) \circ j(w \otimes \mu)\ (x) = j(v \otimes \lambda)(j(w \otimes \mu)(x)) = j(v \otimes \lambda)(\mu(x)\, w)$$

$$= \mu(x)\, j(v \otimes \lambda)(w) = \mu(x)\, \lambda(w)\, v$$

Going the other way around, to the right then up, we obtain the endomorphism which maps x to

$$j(\, c((v \otimes \lambda) \times (w \otimes \mu)))\ (x) = j(\, \lambda(w)(v \otimes \mu)\,)\ (x) = \lambda(w)\, \mu(x)\, v$$

These two outcomes are the same. ///

[19.15] Under the isomorphism of the previous problem, show that the linear map

$$\mathrm{tr}\ : \mathrm{End}_k(V) \to k$$

is the linear map

$$V \otimes V^* \to k$$

induced by the bilinear map $v \times \lambda \to \lambda(v)$.

Note that the induced map

$$V \otimes_k V^* \to k \quad \text{by} \quad v \otimes \lambda \to \lambda(v)$$

is another **contraction map** on tensors. Part of the issue is to compare the coordinate-bound trace with the induced (contraction) map $t(v \otimes \lambda) = \lambda(v)$ determined uniquely from the bilinear map $v \times \lambda \to \lambda(v)$. To this end, let $e_1, \ldots, e_n$ be a basis for V, with dual basis $\lambda_1, \ldots, \lambda_n$. The corresponding matrix coefficients $T_{ij} \in k$ of a k-linear endomorphism T of V are

$$T_{ij} = \lambda_i(Te_j)$$

(Always there is the worry about interchange of the indices.) Thus, in these coordinates,

$$\mathrm{tr}\, T = \sum_i \lambda_i(Te_i)$$

Let $T = j(e_s \otimes \lambda_t)$. Then, since $\lambda_t(e_i) = 0$ unless $i = t$,

$$\mathrm{tr}\, T = \sum_i \lambda_i(Te_i) = \sum_i \lambda_i(j(e_s \otimes \lambda_t)e_i) = \sum_i \lambda_i(\lambda_t(e_i) \cdot e_s) = \lambda_t(\lambda_t(e_t) \cdot e_s) = \begin{cases} 1 & (s = t) \\ 0 & (s \neq t) \end{cases}$$

On the other hand,

$$t(e_s \otimes \lambda_t) = \lambda_t(e_s) = \begin{cases} 1 & (s = t) \\ 0 & (s \neq t) \end{cases}$$

Thus, these two k-linear functionals agree on the monomials, which span, they are equal. ///

[19.16] Prove that $\mathrm{tr}\,(AB) = \mathrm{tr}\,(BA)$ for two endomorphisms of a finite-dimensional vector space V over a field k, with trace defined as just above.

Since the maps

$$\mathrm{End}_k(V) \times \mathrm{End}_k(V) \to k$$

by

$$A \times B \to \mathrm{tr}\,(AB) \quad \text{and/or} \quad A \times B \to \mathrm{tr}\,(BA)$$

are bilinear, it suffices to prove the equality on (images of) monomials $v \otimes \lambda$, since these span the endomophisms over k. Previous examples have converted the issue to one concerning $V_k^{\otimes} V^*$. (We have already shown that the isomorphism $V \otimes_k V^* \approx \mathrm{End}_k(V)$ is converts a *contraction* map on tensors to composition of endomorphisms, and that the trace on tensors defined as another contraction corresponds to the trace of matrices.) Let tr now denote the contraction-map trace on tensors, and (temporarily) write

$$(v \otimes \lambda) \circ (w \otimes \mu) = \lambda(w)\, v \otimes \mu$$

for the contraction-map composition of endomorphisms. Thus, we must show that

$$\mathrm{tr}\ (v \otimes \lambda) \circ (w \otimes \mu) = \mathrm{tr}\ (w \otimes \mu) \circ (v \otimes \lambda)$$

The left-hand side is

$$\mathrm{tr}\ (v \otimes \lambda) \circ (w \otimes \mu) = \mathrm{tr}\,(\,\lambda(w)\, v \otimes \mu) = \lambda(w)\,\mathrm{tr}\,(v \otimes \mu) = \lambda(w)\,\mu(v)$$

The right-hand side is

$$\mathrm{tr}\ (w \otimes \mu) \circ (v \otimes \lambda) = \mathrm{tr}\,(\,\mu(v)\, w \otimes \lambda) = \mu(v)\,\mathrm{tr}\,(w \otimes \lambda) = \mu(v)\,\lambda(w)$$

These elements of k are the same. ///

[19.17] Prove that tensor products are *associative*, in the sense that, for R-modules A, B, C, we have a *natural isomorphism*

$$A \otimes_R (B \otimes_R C) \approx (A \otimes_R B) \otimes_R C$$

In particular, *do* prove the *naturality*, at least the one-third part of it which asserts that, for every R-module homomorphism $f : A \to A'$, the diagram

$$\begin{array}{ccc} A \otimes_R (B \otimes_R C) & \xrightarrow{\approx} & (A \otimes_R B) \otimes_R C \\ \downarrow {\scriptstyle f\otimes(1_B \otimes 1_C)} & & \downarrow {\scriptstyle (f \otimes 1_B)\otimes 1_C} \\ A' \otimes_R (B \otimes_R C) & \xrightarrow{\approx} & (A' \otimes_R B) \otimes_R C \end{array}$$

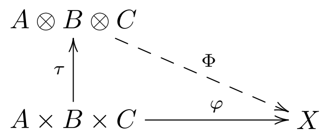
commutes, where the two horizontal isomorphisms are those determined in the first part of the problem. (One might also consider maps $g : B \to B'$ and $h : C \to C'$, but these behave similarly, so there's no real compulsion to worry about them, apart from awareness of the issue.)

Since all tensor products are over R, we drop the subscript, to lighten the notation. As usual, to make a (linear) map *from* a tensor product $M \otimes N$, we induce uniquely from a bilinear map on $M \times N$. We have done this enough times that we will suppress this part now.

The thing that is slightly less trivial is construction of maps *to* tensor products $M \otimes N$. These are always obtained by composition with the canonical bilinear map

$$M \times N \to M \otimes N$$

Important at present is that we can create n-fold tensor products, as well. Thus, we prove the indicated isomorphism by proving that both the indicated iterated tensor products are (naturally) isomorphic to the un-parenthesis'd tensor product $A \otimes B \otimes C$, with canonical map $\tau : A \times B \times C \to A \otimes B \otimes C$, such that for every trilinear map $\varphi : A \times B \times C \to X$ there is a unique linear $\Phi : A \otimes B \otimes C \to X$ such that



The set map

$$A \times B \times C \approx (A \times B) \times C \to (A \otimes B) \otimes C$$

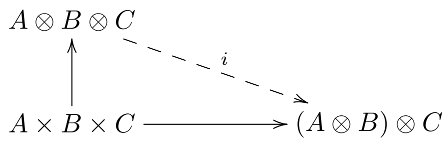
by

$$a \times b \times c \to (a \times b) \times c \to (a \otimes b) \otimes c$$

is linear in each single argument (for fixed values of the others). Thus, we are assured that there is a unique induced linear map

$$A \otimes B \otimes C \to (A \otimes B) \otimes C$$

such that



commutes.

Similarly, from the set map

$$(A \times B) \times C \approx A \times B \times C \to A \otimes B \otimes C$$

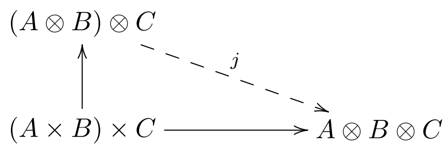
by

$$(a \times b) \times c \to a \times b \times c \to a \otimes b \otimes c$$

is linear in each single argument (for fixed values of the others). Thus, we are assured that there is a unique induced linear map

$$(A \otimes B) \otimes C \to A \otimes B \otimes C$$

such that



commutes.

Then $j \circ i$ is a map of $A \otimes B \otimes C$ to itself compatible with the canonical map $A \times B \times C \to A \otimes B \otimes C$. By uniqueness, $j \circ i$ is the identity on $A \otimes B \otimes C$. Similarly (just very slightly more complicatedly), $i \circ j$ must be the identity on the iterated tensor product. Thus, these two maps are mutual inverses.

To prove naturality in one of the arguments A, B, C, consider $f : C \to C'$. Let j_{ABC} be the isomorphism for a fixed triple A, B, C, as above. The diagram of maps of cartesian products (of sets, at least)

$$\begin{array}{ccc} (A \times B) \times C & \xrightarrow{j_{ABC}} & A \times B \times C \\ \downarrow {\scriptstyle (1_A \times 1_B) \times f} & & \downarrow {\scriptstyle 1_A \times 1_B \times f} \\ (A \times B) \times C & \xrightarrow{j} & A \times B \times C \end{array}$$

does commute: going down, then right, is

$$j_{ABC'}\,((1_A \times 1_B) \times f)((a \times b) \times c)) = j_{ABC'}\,((a \times b) \times f(c)) = a \times b \times f(c)$$

Going right, then down, gives

$$(1_A \times 1_B \times f)\,(j_{ABC}((a \times b) \times c)) = (1_A \times 1_B \times f)\,(a \times b \times c)) = a \times b \times f(c)$$

These are the same. ///