Looking closely, we have a $\Phi(x) = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10} + \ldots$, or that we are computing in formal power series rings. Thus, polynomials are either 1, 0, or 1, based on fairly extensive hand calculations, one might imagine that all coefficients of all cyclotomic polynomials are either $+1$, $-1$, or 0, but this is not true! It is true for $n$ prime, and for $n$ having at most 2 distinct prime factors, but not generally.

The smallest $n$ where $\Phi_n(x)$ has an exotic coefficient is $n = 105$. It is no coincidence that $105 = 3 \cdot 5 \cdot 7$ is the product of the first 3 primes above 2.

Part of the point in finding an exotic coefficient of $\Phi_{105}$ is demonstration that insightful hand calculation can go much further than we might imagine, giving directly-human-verifiable information.

$$\Phi_{105}(x) = \frac{x^{105} - 1}{\Phi_3(x)\Phi_5(x)\Phi_7(x)\Phi_{15}(x)\Phi_{21}(x)\Phi_{35}(x)} = \frac{x^{105} - 1}{\Phi_3(x)\Phi_{15}(x)\Phi_{21}(x)(x^{35} - 1)}$$

$$= \frac{x^{70} + x^{35} + 1}{\Phi_3(x)\Phi_{15}(x)\Phi_{21}(x)} = \frac{(x^{70} + x^{35} + 1)(x^7 - 1)}{\Phi_{15}(x)(x^{21} - 1)} = \frac{(x^{70} + x^{35} + 1)(x^7 - 1)\Phi_1(x)\Phi_3(x)\Phi_5(x)}{(x^{15} - 1)(x^{21} - 1)}$$

$$= \frac{(x^{70} + x^{35} + 1)(x^7 - 1)(x^5 - 1)(x^2 + x + 1)}{(x^{15} - 1)(x^{21} - 1)}$$

Instead of polynomial computations, it suffices to do power series computations, imagining either that $|x| < 1$, or that we are computing in formal power series rings. Thus,

$$\frac{1}{x^{21} - 1} = \frac{-1}{1 - x^{21}} = -(1 + x^{21} + x^{42} + x^{63} + \ldots)$$

The degree of $\Phi_{105}(x)$ is $\varphi(105) = (3 - 1)(5 - 1)(7 - 1) = 48$, and the coefficients of cyclotomic polynomials are the same back-to-front as front-to-back. Thus, in power series in $x$, to hunt for exotic coefficients of $\Phi_{105}$, it suffices to ignore terms of degree above 24. That is, in $\mathbb{Z}[[x]]/\langle x^{25} \rangle$,

$$\Phi_{105}(x) = \frac{(x^{70} + x^{35} + 1)(x^7 - 1)(x^5 - 1)(x^2 + x + 1)}{(x^{15} - 1)(x^{21} - 1)} = (1 + x + x^2)(1 - x^7)(1 - x^5)(1 + x^{15})(1 + x^{21})$$

$$= (1 + x + x^2) \times (1 - x^5 - x^7 + x^{12} + x^{15} - x^{20} + x^{21} - x^{22})$$

$$= 1 + x + x^2 - x^5 - x^6 - x^7 - x^8 - x^9 + x^{12} + x^{13} + x^{14} + x^{15} + x^{16} + x^{17} - x^{20} - x^{21} - x^{22} + x^{23} + x^{24} - x^{25} - x^{26} - x^{27} - x^{28}$$

Looking closely, we have a $-2x^7$. 

\[\text{(February 22, 2024)}\]

\textbf{The 105}^{th} \textbf{ cyclotomic polynomial}

\textit{Paul Garrett}  \ garrett@umn.edu  \ https://www-users.cse.umn.edu/~garrett/