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The 105^{th} *cyclotomic polynomial*

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Based on fairly extensive hand calculations, one might imagine that all coefficients of all cyclotomic polynomials are either $+1$, -1, or 0, but this is not true! It *is* true for n prime, and for n having at most 2 distinct prime factors, but not generally.

The smallest n where $\Phi_n(x)$ has an exotic coefficient is $n = 105$. It is no coincidence that $105 = 3 \cdot 5 \cdot 7$ is the product of the first 3 primes above 2.

Part of the point in finding an exotic coefficient of Φ_{105} is demonstration that insightful hand calculation can go much further than we might imagine, giving directly-human-verifiable information.

$$\Phi_{105}(x) \;=\; \frac{x^{105}-1}{\Phi_1(x)\Phi_3(x)\Phi_5(x)\Phi_7(x)\Phi_{15}(x)\Phi_{21}(x)\Phi_{35}(x)} \;=\; \frac{x^{105}-1}{\Phi_3(x)\Phi_{15}(x)\Phi_{21}(x)(x^{35}-1)}$$

$$= \frac{x^{70}+x^{35}+1}{\Phi_3(x)\Phi_{15}(x)\Phi_{21}(x)} \;=\; \frac{(x^{70}+x^{35}+1)(x^7-1)}{\Phi_{15}(x)(x^{21}-1)} \;=\; \frac{(x^{70}+x^{35}+1)(x^7-1)\Phi_1(x)\Phi_3(x)\Phi_5(x)}{(x^{15}-1)(x^{21}-1)}$$

$$= \frac{(x^{70}+x^{35}+1)(x^7-1)(x^5-1)(x^2+x+1)}{(x^{15}-1)(x^{21}-1)}$$

Instead of polynomial computations, it suffices to do *power series* computations, imagining either that $|x| < 1$, or that we are computing in formal power series rings. Thus,

$$\frac{1}{x^{21}-1} \;=\; \frac{-1}{1-x^{21}} \;=\; -(1+x^{21}+x^{42}+x^{63}+\ldots)$$

The degree of $\Phi_{105}(x)$ is $\varphi(105) = (3-1)(5-1)(7-1) = 48$, and the coefficients of cyclotomic polynomials are the same back-to-front as front-to-back. Thus, in power series in x, to hunt for exotic coefficients of Φ_{105}, it suffices to ignore terms of degree above 24. That is, in $\mathbb{Z}[[x]]/\langle x^{25}\rangle$,

$$\Phi_{105}(x) \;=\; \frac{(x^{70}+x^{35}+1)(x^7-1)(x^5-1)(x^2+x+1)}{(x^{15}-1)(x^{21}-1)} \;=\; (1+x+x^2)(1-x^7)(1-x^5)(1+x^{15})(1+x^{21})$$

$$= (1+x+x^2) \times (1-x^5-x^7+x^{12}+x^{15}-x^{20}+x^{21}-x^{22})$$

$$=1+x+x^2-x^5-x^6-x^7-x^7-x^8-x^9+x^{12}+x^{13}+x^{14}+x^{15}+x^{16}+x^{17}-x^{20}-x^{21}-x^{22}+x^{21}+x^{22}+x^{23}-x^{22}-x^{23}-x^{24}$$

$$= 1+x+x^2-x^5-x^6-2x^7-x^8-x^9+x^{12}+x^{13}+x^{14}+x^{15}+x^{16}+x^{17}-x^{20}-x^{22}-x^{24}$$

Looking closely, we have a $-2x^7$.
