27. Tensor products

In this first pass at tensor products, we will only consider tensor products of modules over commutative rings with identity. This is not at all a critical restriction, but does offer many simplifications, while still illuminating many important features of tensor products and their applications.

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1. *Desiderata*

It is time to take stock of what we are missing in our development of linear algebra and related matters.

Most recently, we are missing the proof of existence of determinants, although linear algebra is sufficient to give palatable proofs of the properties of determinants.

We want to be able to give a direct and natural proof of the Cayley-Hamilton theorem (without using the structure theorem for finitely-generated modules over PIDs). This example suggests that linear algebra over *fields* is insufficient.

We want a sufficient conceptual situation to be able to finish the *uniqueness* part of the structure theorem for finitely-generated modules over PIDs. Again, linear or multi-linear algebra over fields is surely insufficient for this.

We might want an antidote to the antique styles of discussion of *vectors* v_i [sic], *covectors* v^i [sic], *mixed tensors* T_k^{ij}, and other vague entities whose nature was supposedly specified by the number and pattern of upper and lower subscripts. These often-ill-defined notions came into existence in the mid-to-late 19th century in the development of geometry. Perhaps the impressive point is that, even without adequate algebraic grounding, people managed to envision and roughly formulate geometric notions.

In a related vein, at the beginning of calculus of several variables, one finds ill-defined notions and ill-made distinctions between

$$dx\,dy$$

and

$$dx \wedge dy$$

with the nature of the so-called *differentials* dx and dy even less clear. For a usually unspecified reason,

$$dx \wedge dy = -dy \wedge dx$$

though perhaps

$$dx\,dy = dy\,dx$$

In other contexts, one may find confusion between the integration of *differential forms* versus integration with respect to a *measure*. We will not resolve *all* these confusions here, only the question of what $a \wedge b$ might mean.

Even in fairly concrete linear algebra, the question of **extension of scalars** to convert a real vector space to a complex vector space is possibly mysterious. On one hand, if we are content to say that vectors are *column* vectors or *row* vectors, then we might be equally content in allowing complex entries. For that matter, once a basis for a real vector space is chosen, to write apparent linear combinations with complex coefficients (rather than merely real coefficients) is easy, as symbol manipulation. However, it is quite unclear what meaning can be attached to such expressions. Further, it is unclear what effect a different choice of basis might have on this process. Finally, without a choice of basis, these *ad hoc* notions of extension of scalars are stymied. Instead, the construction below of the tensor product

$$V \otimes_R \mathbb{C} = \text{complexification of } V$$

of a real vectorspace V with $\mathbb{C}$ over $\mathbb{R}$ is exactly right, as will be discussed later.

The notion of *extension of scalars* has important senses in situations which are qualitatively different than complexification of real vector spaces. For example, there are several reasons to want to convert *abelian groups* A ($\mathbb{Z}$-modules) into $\mathbb{Q}$-vectorspaces in some reasonable, natural manner. After explicating a minimalist notion of reasonability, we will see that a tensor product

$$A \otimes_{\mathbb{Z}} \mathbb{Q}$$

is just right.

There are many examples of application of the construction and universal properties of tensor products.

2. *Definitions, uniqueness, existence*

Let R be a commutative ring with 1. We will only consider R-modules M with the property[1] that $1 \cdot m = m$ for all $m \in M$. Let M, N, and X be R-modules. A map

$$B : M \times N \longrightarrow X$$

is R-**bilinear** if it is R-linear separately in each argument, that is, if

$$\begin{array}{rcl} B(m+m',n) &=& B(m,n)+B(m',n) \\ B(rm,n) &=& r \cdot B(m,n) \\ B(m,n+n') &=& B(m,n)+B(m,n') \\ B(m,rn) &=& r \cdot B(m,n) \end{array}$$

for all $m, m' \in M$, $n, n' \in N$, and $r \in R$.

As in earlier discussion of free modules, and in discussion of polynomial rings as free algebras, we will define tensor products by *mapping properties*. This will allow us an easy proof that tensor products (if they exist) are *unique* up to *unique isomorphism*. Thus, whatever construction we contrive must inevitably yield the same (or, better, *equivalent*) object. Then we give a modern construction.

A **tensor product** of R-modules M, N is an R-module denoted $M \otimes_R N$ together with an R-bilinear map

$$\tau : M \times N \longrightarrow M \otimes_R N$$

such that, for every R-bilinear map

$$\varphi : M \times N \longrightarrow X$$

there is a unique *linear* map

$$\Phi : M \otimes_{\mathbb{R}} N \longrightarrow X$$

such that the diagram

$$\begin{tikzcd} M \otimes_R N \arrow[dr, dashed, "\Phi"] & \\ M \times N \arrow[u, "\tau"] \arrow[r, "\varphi"'] & X \end{tikzcd}$$

commutes, that is, $\varphi = \Phi \circ \tau$.

The usual notation does not involve any symbol such as τ, but, rather, denotes the image $\tau(m \times n)$ of $m \times n$ in the tensor product by

$$m \otimes n = \text{image of } m \times n \text{ in } M \otimes_R N$$

In practice, the implied R-bilinear map

$$M \times N \longrightarrow M \otimes_R N$$

is often left anonymous. This seldom causes serious problems, but we will be temporarily more careful about this while setting things up and proving basic properties.

The following proposition is typical of uniqueness proofs for objects defined by mapping property requirements. Note that internal details of the objects involved play no role. Rather, the argument proceeds by manipulation of arrows.

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[1] Sometimes such a module M is said to be **unital**, but this terminology is not universal, and, thus, somewhat unreliable. Certainly the term is readily confused with other usages.

[2.0.1] **Proposition:** Tensor products $M \otimes_R N$ are unique up to unique isomorphism. That is, given two tensor products

$$\tau_1 : M \times N \longrightarrow T_1$$

$$\tau_2 : M \times N \longrightarrow T_2$$

there is a *unique isomorphism* $i : T_1 \longrightarrow T_2$ such that the diagram

$$\begin{tikzcd} & T_1 \arrow[dd, dashed, "i"] \\ M \times N \arrow[ur, "\tau_1"] \arrow[dr, "\tau_2"'] & \\ & T_2 \end{tikzcd}$$

commutes, that is, $\tau_2 = i \circ \tau_1$.

Proof: First, we show that for a tensor product $\tau : M \times N \longrightarrow T$, the only map $f : T \longrightarrow T$ compatible with τ is the identity. That is the identity map is the only map f such that

$$\begin{tikzcd} & T \arrow[dd, dashed, "f"] \\ M \times N \arrow[ur, "\tau"] \arrow[dr, "\tau"'] & \\ & T \end{tikzcd}$$

commutes. Indeed, the definition of a tensor product demands that, given the bilinear map

$$\tau : M \times N \longrightarrow T$$

(with T in the place of the earlier X) there is a unique linear map $\Phi : T \longrightarrow T$ such that the diagram

$$\begin{tikzcd} T \arrow[dr, dashed, "\Phi"] & \\ M \times N \arrow[u, "\tau"] \arrow[r, "\tau"'] & T \end{tikzcd}$$

commutes. The identity map on T certainly has this property, so is the *only* map $T \longrightarrow T$ with this property.

Looking at two tensor products, first take $\tau_2 : M \times N \longrightarrow T_2$ in place of the $\varphi : M \times N \longrightarrow X$. That is, there is a unique linear $\Phi_1 : T_1 \longrightarrow T_2$ such that

$$\begin{tikzcd} T_1 \arrow[dr, dashed, "\Phi_1"] & \\ M \times N \arrow[u, "\tau_1"] \arrow[r, "\tau_2"'] & T_2 \end{tikzcd}$$

commutes. Similarly, reversing the roles, there is a unique linear $\Phi_2 : T_2 \longrightarrow T_1$ such that

$$\begin{tikzcd} T_2 \arrow[dr, dashed, "\Phi_2"] & \\ M \times N \arrow[u, "\tau_2"] \arrow[r, "\tau_1"'] & T_1 \end{tikzcd}$$

commutes. Then $\Phi_2 \circ \Phi_1 : T_1 \longrightarrow T_1$ is compatible with τ_1, so is the identity, from the first part of the proof. And, symmetrically, $\Phi_1 \circ \Phi_2 : T_2 \longrightarrow T_2$ is compatible with τ_2, so is the identity. Thus, the maps Φ_i are mutual inverses, so are isomorphisms. ///

For existence, we will give an argument in what might be viewed as an *extravagant* modern style. Its extravagance is similar to that in E. Artin's proof of the existence of algebraic closures of fields, in which we create an indeterminate for each irreducible polynomial, and look at the polynomial ring in these myriad indeterminates. In a similar spirit, the tensor product $M \otimes_R N$ will be created as a quotient of a truly huge module by an only slightly less-huge module.

[2.0.2] **Proposition:** Tensor products $M \otimes_R N$ exist.

Proof: Let $i : M \times N \longrightarrow F$ be the free R-module on the *set* $M \times N$. Let Y be the R-submodule generated by all elements

$$\begin{array}{l} i(m+m',n) - i(m,n) - i(m',n) \\ i(rm,n) - r \cdot i(m,n) \\ i(m,n+n') - i(m,n) - i(m,n') \\ i(m,rn) - r \cdot i(m,n) \end{array}$$

for all $r \in R$, $m, m' \in M$, and $n, n' \in N$. Let

$$q : F \longrightarrow F/Y$$

be the quotient map. We claim that $\tau = q \circ i : M \times N \longrightarrow F/Y$ is a tensor product.

Given a bilinear map $\varphi : M \times N \longrightarrow X$, by properties of free modules there is a unique $\Psi : F \longrightarrow X$ such that the diagram

$$\begin{tikzcd} F \arrow[drr, dashed, "\Psi"] & & \\ M \times N \arrow[u, "i"] \arrow[rr, "\varphi"'] & & X \end{tikzcd}$$

commutes. We claim that Ψ factors through F/Y, that is, that there is $\Phi : F/Y \longrightarrow X$ such that

$$\Psi = \Phi \circ q : F \longrightarrow X$$

Indeed, since $\varphi : M \times N \longrightarrow X$ is bilinear, we conclude that, for example,

$$\varphi(m+m',n) = \varphi(m,n) + \varphi(m',n)$$

Thus,

$$(\Psi \circ i)(m+m',n) = (\Psi \circ i)(m,n) + (\Psi \circ i)(m',n)$$

Thus, since Ψ is linear,

$$\Psi(\ i(m+m',n) - i(m,n) - i(m',n)\ ) = 0$$

A similar argument applies to all the generators of the submodule Y of F, so Ψ does factor through F/Y. Let Φ be the map such that $\Psi = \Phi \circ q$.

A similar argument on the generators for Y shows that the composite

$$\tau = q \circ i : M \times N \longrightarrow F/Y$$

is bilinear, even though i was only a set map.

The uniqueness of Ψ yields the uniqueness of Φ, since q is a surjection, as follows. For two maps Φ_1 and Φ_2 with

$$\Phi_1 \circ q = \Psi = \Phi_2 \circ q$$

given $x \in F/Y$ let $y \in F$ be such that $q(y) = x$. Then

$$\Phi_1(x) = (\Phi_1 \circ q)(y) = \Psi(y) = (\Phi_2 \circ q)(y) = \Phi_2(x)$$

Thus, $\Phi_1 = \Phi_2$. ///

[2.0.3] **Remark:** It is worthwhile to contemplate the many things we did *not* do to prove the uniqueness and the existence.

Lest anyone think that tensor products $M \otimes_R N$ contain anything not implicitly determined by the behavior of the **monomial tensors**[2] $m \otimes n$, we prove

[2.0.4] **Proposition:** The monomial tensors $m \otimes n$ (for $m \in M$ and $n \in N$) generate $M \otimes_R N$ as an R-module.

Proof: Let X be the submodule of $M \otimes_R N$ generated by the monomial tensors, $Q = M \otimes_R N)/X$ the quotient, and $q : M \otimes_R N \longrightarrow Q$ the quotient map. Let

$$B : M \times N \longrightarrow Q$$

be the 0-map. A defining property of the tensor product is that there is a unique R-linear

$$\beta : M \otimes_R N \longrightarrow Q$$

making the usual diagram commute, that is, such that $B = \beta \circ \tau$, where $\tau : M \times N \longrightarrow M \otimes_R N$. Both the quotient map q and the 0-map $M \otimes_R N \longrightarrow Q$ allow the 0-map $M \times N \longrightarrow Q$ to factor through, so by the uniqueness the quotient map is the 0-map. That is, Q is the 0-module, so $X = M \otimes_R N$. ///

[2.0.5] **Remark:** Similarly, define the tensor product

$$\tau : M_1 \times \ldots \times M_n \longrightarrow M_1 \otimes_R \ldots \otimes_R M_n$$

of an arbitrary finite number of R-modules as an R-module and multilinear map τ such that, for any R-multilinear map

$$\varphi : M_1 \times M_2 \times \ldots \times M_n \longrightarrow X$$

there is a unique R-linear map

$$\Phi : M_1 \otimes_R M_2 \otimes_R \ldots \otimes_R M_n \longrightarrow X$$

such that $\varphi = \Phi \circ \tau$. That is, the diagram

$$\begin{tikzcd}
M_1 \otimes_R M_2 \otimes_R \ldots \otimes_R M_n \arrow[dr, dashed, "\Phi"] & \\
M_1 \times M_2 \times \ldots \times M_n \arrow[u, "\tau"] \arrow[r, "\varphi"] & X
\end{tikzcd}$$

commutes. There is the subordinate issue of proving **associativity**, namely, that there are natural isomorphisms

$$(M_1 \otimes_R \ldots \otimes_R M_{n-1}) \otimes_R M_n \approx M_1 \otimes_R (M_2 \otimes_R \ldots \otimes_R M_n)$$

to be sure that we need not worry about parentheses.

3. *First examples*

[2] Again, $m \otimes n$ is the image of $m \times n \in M \times N$ in $M \otimes_R N$ under the map $\tau : M \times N \longrightarrow M \otimes_R N$.

We want to illustrate the possibility of computing [3] tensor products without needing to make any use of any suppositions about the *internal* structure of tensor products.

First, we emphasize that to show that a tensor product $M \otimes_R N$ of two R-modules (where R is a commutative ring with identity) is 0, it suffices to show that all monomial tensors are 0, since these generate the tensor product (as R-module). [4]

Second, we emphasize [5] that in $M \otimes_R N$, with $r \in R$, $m \in M$, and $n \in N$, we can always rearrange

$$(rm) \otimes n = r(m \otimes n) = m \otimes (rn)$$

Also, for $r, s \in R$,

$$(r+s)(m \otimes n) = rm \otimes n + sm \otimes n$$

[3.0.1] Example: Let's experiment [6] first with something like

$$\mathbb{Z}/5 \otimes_{\mathbb{Z}} \mathbb{Z}/7$$

Even a novice may anticipate that the fact that 5 annihilates the left factor, while 7 annihilates the right factor, creates an interesting dramatic tension. What will come of this? For any $m \in \mathbb{Z}/5$ and $n \in \mathbb{Z}/7$, we can do things like

$$0 = 0 \cdot (m \otimes n) = (0 \cdot m) \otimes n = (5 \cdot m) \otimes n = m \otimes 5n$$

and

$$0 = 0 \cdot (m \otimes n) = m \otimes (0 \cdot n) = m \otimes (7 \cdot n) = 7m \otimes n = 2m \otimes n$$

Then

$$(5m \otimes n) - 2 \cdot (2m \otimes n) = (5 - 2 \cdot 2)m \otimes n = m \otimes n$$

but also

$$(5m \otimes n) - 2 \cdot (2m \otimes n) = 0 - 2 \cdot 0 = 0$$

That is, every monomial tensor in $\mathbb{Z}/5 \otimes_{\mathbb{Z}} \mathbb{Z}/7$ is 0, so the whole tensor product is 0.

[3.0.2] Example: More systematically, given relatively prime integers [7] a, b, we claim that

$$\mathbb{Z}/a \otimes_{\mathbb{Z}} \mathbb{Z}/b = 0$$

Indeed, using the Euclidean-ness of $\mathbb{Z}$, let $r, s \in \mathbb{Z}$ such that

$$1 = ra + sb$$

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[3] Of course, it is unclear in what *sense* we are computing. In the simpler examples the tensor product is the 0 module, which needs no further explanation. However, in other cases, we will see that a certain tensor product is the *right answer* to a natural question, without necessarily determining what the tensor product is in some *other* sense.

[4] This was proven just above.

[5] These are merely translations into this notation of part of the definition of the tensor product, but deserve emphasis.

[6] Or pretend, disingenuously, that we don't know what will happen? Still, some tangible numerical examples are worthwhile, much as a picture may be worth many words.

[7] The same argument obviously works as stated in Euclidean rings R, rather than just $\mathbb{Z}$. Further, a restated form works for arbitrary commutative rings R with identity: given two ring elements a, b such that the ideal $Ra + Rb$ generated by *both* is the whole ring, we have $R/a \otimes_R R/b = 0$. The point is that this adjusted hypothesis again gives us $r, s \in R$ such that $1 = ra + sb$, and then the same argument works.

Then

$$m \otimes n = 1 \cdot (m \otimes n) = (ra + sb) \cdot (m \otimes n) = ra(m \otimes n) + s$$

$$= b(m \otimes n) = a(rm \otimes n) + b(m \otimes sn) = a \cdot 0 + b \cdot 0 = 0$$

Thus, every monomial tensor is 0, so the whole tensor product is 0.

[3.0.3] **Remark:** Yes, it somehow not visible that these should be 0, since we probably think of tensors are complicated objects, not likely to be 0. But this vanishing is an assertion that there are *no* non-zero $\mathbb{Z}$-bilinear maps from $\mathbb{Z}/5 \times \mathbb{Z}/7$, which is a plausible more-structural assertion.

[3.0.4] **Example:** Refining the previous example: let a, b be arbitrary non-zero integers. We claim that

$$\mathbb{Z}/a \otimes_{\mathbb{Z}} \mathbb{Z}/b \approx \mathbb{Z}/\gcd(a,b)$$

First, take $r, s \in \mathbb{Z}$ such that

$$\gcd(a,b) = ra + sb$$

Then the same argument as above shows that this *gcd* annihilates every monomial

$$(ra + sb)(m \otimes n) = r(am \otimes n) + s(m \otimes bn) = r \cdot 0 + s \cdot 0 = 0$$

Unlike the previous example, we are not entirely done, since we didn't simply prove that the tensor product is 0. We need something like

[3.0.5] **Proposition:** Let $\{m_\alpha : \alpha \in A\}$ be a set of generators for an R-module M, and $\{n_\beta : \beta \in B\}$ a set of generators for an R-module N. Then

$$\{m_\alpha \otimes n_\beta : \alpha \in A,\ \beta \in B\}$$

is a set of generators [8] for $M \otimes_R N$.

Proof: Since monomial tensors generate the tensor product, it suffices to show that every monomial tensor is expressible in terms of the $m_\alpha \otimes n_\beta$. Unsurprisingly, taking r_α and s_β in R (0 for all but finitely-many indices), by multilinearity

$$(\sum_\alpha r_\alpha m_\alpha) \otimes (\sum_\beta s_\beta n_\beta) = \sum_{\alpha,\beta} r_\alpha s_\beta \; m_\alpha \otimes n_\beta$$

This proves that the special monomials $m_\alpha \otimes n_\beta$ generate the tensor product. ///

Returning to the example, since $1 + a\mathbb{Z}$ generates $\mathbb{Z}/a$ and $1 + b\mathbb{Z}$ generates $\mathbb{Z}/b$, the proposition assures us that $1 \otimes 1$ generates the tensor product. We already know that

$$\gcd(a,b) \cdot 1 \otimes 1 = 0$$

Thus, we know that $\mathbb{Z}/a \otimes \mathbb{Z}/b$ is isomorphic to *some quotient of* $\mathbb{Z}/\gcd(a,b)$.

But this does not preclude the possibility that something else is 0 for a reason we didn't anticipate. One more ingredient is needed to prove the claim, namely exhibition of a sufficiently non-trivial bilinear map to eliminate the possibility of any further collapsing. One might naturally contrive a $\mathbb{Z}$-blinear map with formulaic expression

$$B(x,y) = xy \ldots$$

[8] It would be unwise, and generally very difficult, to try to give generators *and relations* for tensor products.

but there may be some difficulty in intuiting where that xy resides. To understand this, we must be scrupulous about cosets, namely

$$(x + a\mathbb{Z}) \cdot (y + b\mathbb{Z}) = xy + ay\mathbb{Z} + bx\mathbb{Z} + ab\mathbb{Z} \subset xy + a\mathbb{Z} + b\mathbb{Z} = xy + \gcd(a,b)\mathbb{Z}$$

That is, the bilinear map is

$$B : \mathbb{Z}/a \times \mathbb{Z}/b \longrightarrow \mathbb{Z}/\gcd(a,b)$$

By construction,

$$B(1,1) = 1 \in \mathbb{Z}/\gcd(a,b)$$

so

$$\beta(1 \otimes 1) = B(1,1) = 1 \in \mathbb{Z}/\gcd(a,b)$$

In particular, the map is a surjection. Thus, knowing that the tensor product is generated by $1 \otimes 1$, and that this element has order *dividing* $\gcd(a,b)$, we find that it has order *exactly* $\gcd(a,b)$, so is *isomorphic* to $\mathbb{Z}/\gcd(a,b)$, by the map

$$x \otimes y \longrightarrow xy$$

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4. *Tensor products $f \otimes g$ of maps*

Still R is a commutative ring with 1.

An important type of map on a tensor product arises from pairs of R-linear maps on the modules in the tensor product. That is, let

$$f : M \longrightarrow M' \qquad g : N \longrightarrow N'$$

be R-module maps, and attempt to define

$$f \otimes g : M \otimes_R N \longrightarrow M' \otimes_R N'$$

by

$$(f \otimes g)(m \otimes n) = f(m) \otimes g(n)$$

Justifiably interested in being sure that this formula makes sense, we proceed as follows.

If the map is *well-defined* then it is defined completely by its values on the monomial tensors, since these generate the tensor product. To prove well-definedness, we invoke the defining property of the tensor product, by first considering a bilinear map

$$B : M \times N \longrightarrow M' \otimes_R N'$$

given by

$$B(m \times n) = f(m) \otimes g(n)$$

To see that *this* bilinear map is well-defined, let

$$\tau' : M' \times N' \longrightarrow M' \otimes_R N'$$

For fixed $n \in N$, the composite

$$m \longrightarrow f(m) \longrightarrow \tau'(f(m), g(n)) = f(m) \otimes g(n)$$

is certainly an R-linear map in m. Similarly, for fixed $m \in M$,

$$n \longrightarrow g(n) \longrightarrow \tau'(f(m), g(n)) = f(m) \otimes g(n)$$

is an R-linear map in n. Thus, B is an R-bilinear map, and the formula for $f \otimes g$ expresses the induced linear map on the tensor product.

Similarly, for an n-tuple of R-linear maps

$$f_i : M_i \longrightarrow N_i$$

there is an associated linear

$$f_1 \otimes \ldots \otimes f_n : M_1 \otimes \ldots \otimes M_n \longrightarrow N_1 \otimes \ldots \otimes N_n$$

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5. *Extension of scalars, functoriality, naturality*

How to turn an R-module M into an S-module? [9] We assume that R and S are commutative rings with unit, and that there is a ring homomorphism $\alpha : R \longrightarrow S$ such that $\alpha(1_R) = 1_S$. For example the situation that $R \subset S$ with $1_R = 1_S$ is included. But also we want to allow not-injective maps, such as quotient maps $\mathbb{Z} \longrightarrow \mathbb{Z}/n$. This makes S an R-algebra, by

$$r \cdot s = \alpha(r)s$$

Before describing the *internal details* of this conversion, we should tell what criteria it should meet. Let

$$F : \{R - \text{modules}\} \longrightarrow \{S - \text{modules}\}$$

be this conversion. [10] Our main requirement is that for R-modules M and S-modules N, there should be a natural [11] isomorphism [12]

$$\mathrm{Hom}_S(FM, N) \approx \mathrm{Hom}_R(M, N)$$

where on the right side we *forget* that N is an S-module, remembering only the action of R on it. If we want to make more explicit this *forgetting*, we can write

$$\mathrm{Res}^S_R N = R\text{-module obtained by forgetting } S\text{-module structure on } N$$

and then, more carefully, write what we want for extension of scalars as

$$\mathrm{Hom}_S(FM, N) \approx \mathrm{Hom}_R(M, \mathrm{Res}^S_R N)$$

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[9] As an alert reader can guess, the anticipated answer involves tensor products. However, we can lend some dignity to the proceedings by explaining *requirements* that should be met, rather than merely contriving from an R-module a thing that happens to be an S-module.

[10] This F would be an example of a **functor** from the **category** of R-modules and R-module maps to the **category** of S-modules and S-module maps. To be a genuine functor, we should also tell how F converts R-module *homomorphisms* to S-module homomorphisms. We do not need to develop the formalities of category theory just now, so will not do so. In fact, direct development of a variety of such examples surely provides the only sensible and genuine motivation for a later formal development of category theory.

[11] This sense of *natural* will be made precise shortly. It is the same sort of naturality as discussed earlier in the simplest example of second duals of finite-dimensional vector spaces over fields.

[12] It suffices to consider the map as an isomorphism of abelian groups, but, in fact, the isomorphism potentially makes sense as an S-module map, if we give both sides S-module structures. For $\Phi \in \mathrm{Hom}_S(FM, N)$, there is an unambiguous and unsurprising S-module structure, namely $(s\Phi)(m') = s \cdot \Phi(m') = s \cdot \Phi(m')$ for $m' \in FM$ and $s \in S$. For $\varphi \in \mathrm{Hom}_R(M, N)$, since N does have the additional structure of S-module, we have $(s \cdot \varphi)(m) = s \cdot \varphi(m)$.

Though we'll not use it much in the immediate sequel, this extra notation does have the virtue that it makes clear that *something happened* to the module N.

This association of an S-module FM to an R-module M is *not* itself a module map. Instead, it is a **functor** from R-modules to S-modules, meaning that for an R-module map $f : M \longrightarrow M'$ there should be a naturally associated S-module map $Ff : FM \longrightarrow FM'$. Further, F should respect the composition of module homomorphisms, namely, for R-module homomorphisms

$$M \xrightarrow{f} M' \xrightarrow{g} M''$$

it should be that

$$F(g \circ f) = Fg \circ Ff : FM \longrightarrow FM''$$

This already makes clear that we shouldn't be completely cavalier in converting R-modules to S-modules.

Now we are able to describe the **naturality** we require of the desired isomorphism

$$\mathrm{Hom}_S(FM, N) \xrightarrow{i_{M,N}} \mathrm{Hom}_R(M, N)$$

One part of the naturality is **functoriality in** N, which requires that for every R-module map $g : N \longrightarrow N'$ the diagram

$$\begin{tikzcd}
\mathrm{Hom}_S(FM, N) \arrow[r, "i_{M,N}"] \arrow[d, "g\circ-"] & \mathrm{Hom}_R(M, N) \arrow[d, "g\circ-"] \\
\mathrm{Hom}_S(FM, N') \arrow[r, "i_{M,N'}"] & \mathrm{Hom}_R(M, N')
\end{tikzcd}$$

commutes, where the map $g \circ -$ is (post-) composition with g, by

$$g \circ - \ : \varphi \longrightarrow g \circ \varphi$$

Obviously one oughtn't imagine that it is easy to haphazardly guess a functor F possessing such virtues. [13] There is also the requirement of **functoriality in** M, which requires for every $f : M \longrightarrow M'$ that the diagram

$$\begin{tikzcd}
\mathrm{Hom}_S(FM, N) \arrow[r, "i_{M,N}"] & \mathrm{Hom}_R(M, N) \\
\mathrm{Hom}_S(FM', N) \arrow[r, "i_{M',N}"] \arrow[u, "-\circ Ff"] & \mathrm{Hom}_R(M', N) \arrow[u, "-\circ f"']
\end{tikzcd}$$

commutes, where the map $- \circ Ff$ is (pre-) composition with Ff, by

$$- \circ Ff \ : \varphi \longrightarrow \varphi \circ Ff$$

After all these demands, it is a relief to have

[5.0.1] Theorem: The extension-of-scalars (from R to S) module FM attached to an R-module M is

$$\text{extension-of-scalars-}R\text{-to-}S \text{ of } \ M = M \otimes_R S$$

[13] Further, the same attitude might demand that we worry about the *uniqueness* of such F. Indeed, there is such a uniqueness statement that can be made, but more preparation would be required than we can afford just now. The assertion would be about uniqueness of *adjoint functors*.

That is, for every R-module M and S-module N there is a *natural* isomorphism

$$\mathrm{Hom}_S(M \otimes_R S, N) \xrightarrow{i_{M,N}} \mathrm{Hom}_R(M, \mathrm{Res}^S_R N)$$

given by

$$i_{M,N}(\Phi)(m) = \Phi(m \otimes 1)$$

for $\Phi \in \mathrm{Hom}_S(M \otimes_R S, N)$, with inverse

$$j_{M,N}(\varphi)(m \otimes s) = s \cdot \varphi(m)$$

for $s \in S$, $m \in M$.

Proof: First, we verify that the map $i_{M,N}$ given in the statement is an isomorphism, and then prove the functoriality in N, and functoriality in M.

For the moment, write simply i for $i_{M,N}$ and j for $j_{M,N}$. Then

$$((j \circ i)\Phi)(m \otimes s) = (j(i\Phi))(m \otimes s) = s \cdot (i\Phi)(m) = s \cdot \Phi(m \otimes 1) = \Phi(m \otimes s)$$

and

$$((i \circ j)\varphi)(m) = (i(j\varphi))(m) = (j\varphi)(m \otimes 1) = 1 \cdot \varphi(m) = \varphi(m)$$

This proves that the maps are isomorphisms.

For functoriality in N, we must prove that for every R-module map $g : N \longrightarrow N'$ the diagram

$$\begin{tikzcd}
\mathrm{Hom}_S(M \otimes_R S, N) \arrow[r, "i_{M,N}"] \arrow[d, "g\circ-"] & \mathrm{Hom}_R(M, N) \arrow[d, "g\circ-"] \\
\mathrm{Hom}_S(M \otimes_R S, N') \arrow[r, "i_{M,N'}"] & \mathrm{Hom}_R(M, N')
\end{tikzcd}$$

commutes. For brevity, let $i = i_{M,N}$ and $i' = i_{M,N'}$. Directly computing, using the definitions,

$$((i' \circ (g \circ -))\Phi)(m) = (i' \circ (g \circ \Phi))(m) = (g \circ \Phi)(m \otimes 1)$$

$$= g(\Phi(m \otimes 1)) = g(i\Phi(m)) = ((g \circ -) \circ i)\Phi)(m)$$

For functoriality in M, for each R-module homomorphism $f : M \longrightarrow M'$ we must prove that the diagram

$$\begin{tikzcd}
\mathrm{Hom}_S(M \otimes_R S, N) \arrow[r, "i_{M,N}"] & \mathrm{Hom}_R(M, N) \\
\mathrm{Hom}_S(M' \otimes_R S, N) \arrow[r, "i_{M',N}"] \arrow[u, "-\circ Ff"] & \mathrm{Hom}_R(M', N) \arrow[u, "-\circ f"']
\end{tikzcd}$$

commutes, where $f \otimes 1$ is the map of $M \otimes_R S$ to itself determined by

$$(f \otimes 1)(m \otimes s) = f(m) \otimes s$$

and $- \circ (f \otimes 1)$ is (pre-) composition with this function. Again, let $i = i_{M,N}$ and $i' = i_{M',N}$, and compute directly

$$(((- \circ f) \circ i')\Psi)(m) = ((- \circ f)(i'\Psi)(m) = (i'\Psi \circ f)(m) = (i'\Psi)(fm)$$

$$= \Psi(fm \otimes 1) = (\Psi \circ (f \otimes 1))(m \otimes 1) = (i(\Psi \circ (f \otimes 1)))(m) = ((i \circ (- \circ (f \otimes 1)))\Psi)(m)$$

Despite the thicket of parentheses, this does prove what we want, namely, that

$$(- \circ f) \circ i' = i \circ (- \circ (f \otimes 1))$$

proving the functoriality of the isomorphism in M. ///

6. *Worked examples*

[27.1] For distinct primes p, q, compute

$$\mathbb{Z}/p \otimes_{\mathbb{Z}/pq} \mathbb{Z}/q$$

where for a divisor d of an integer n the abelian group $\mathbb{Z}/d$ is given the $\mathbb{Z}/n$-module structure by

$$(r + n\mathbb{Z}) \cdot (x + d\mathbb{Z}) = rx + d\mathbb{Z}$$

We claim that this tensor product is 0. To prove this, it suffices to prove that every $m \otimes n$ (the image of $m \times n$ in the tensor product) is 0, since we have shown that these *monomial* tensors always generate the tensor product.

Since p and q are relatively prime, there exist integers a, b such that $1 = ap + bq$. Then for all $m \in \mathbb{Z}/p$ and $n \in \mathbb{Z}/q$,

$$m \otimes n = 1 \cdot (m \otimes n) = (ap + bq)(m \otimes n) = a(pm \otimes n) + b(m \otimes qn) = a \cdot 0 + b \cdot 0 = 0$$

An auxiliary point is to recognize that, indeed, $\mathbb{Z}/p$ and $\mathbb{Z}/q$ really are $\mathbb{Z}/pq$-modules, and that the equation $1 = ap + bq$ still does make sense inside $\mathbb{Z}/pq$. ///

[27.2] Compute $\mathbb{Z}/n \otimes_{\mathbb{Z}} \mathbb{Q}$ with $0 < n \in \mathbb{Z}$.

We claim that the tensor product is 0. It suffices to show that every $m \otimes n$ is 0, since these monomials generate the tensor product. For any $x \in \mathbb{Z}/n$ and $y \in \mathbb{Q}$,

$$x \otimes y = x \otimes (n \cdot \frac{y}{n}) = (nx) \otimes \frac{y}{n} = 0 \otimes \frac{y}{n} = 0$$

as claimed. ///

[27.3] Compute $\mathbb{Z}/n \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}$ with $0 < n \in \mathbb{Z}$.

We claim that the tensor product is 0. It suffices to show that every $m \otimes n$ is 0, since these monomials generate the tensor product. For any $x \in \mathbb{Z}/n$ and $y \in \mathbb{Q}/\mathbb{Z}$,

$$x \otimes y = x \otimes (n \cdot \frac{y}{n}) = (nx) \otimes \frac{y}{n} = 0 \otimes \frac{y}{n} = 0$$

as claimed. ///

[27.4] Compute $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Q}/\mathbb{Z})$ for $0 < n \in \mathbb{Z}$.

Let $q : \mathbb{Z} \longrightarrow \mathbb{Z}/n$ be the natural quotient map. Given $\varphi \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Q}/\mathbb{Z})$, the composite $\varphi \circ q$ is a $\mathbb{Z}$-homomorphism from the free $\mathbb{Z}$-module $\mathbb{Z}$ (on one generator 1) to $\mathbb{Q}/\mathbb{Z}$. A homomorphism $\Phi \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$ is completely determined by the image of 1 (since $\Phi(\ell) = \Phi(\ell \cdot 1) = \ell \cdot \Phi(1)$), and since $\mathbb{Z}$ is *free* this image can be *anything* in the target $\mathbb{Q}/\mathbb{Z}$.

Such a homomorphism $\Phi \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$ factors through $\mathbb{Z}/n$ if and only if $\Phi(n) = 0$, that is, $n \cdot \Phi(1) = 0$. A complete list of representatives for equivalence classes in $\mathbb{Q}/\mathbb{Z}$ annihilated by n is $0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \ldots, \frac{n-1}{n}$. Thus, $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Q}/\mathbb{Z})$ is in bijection with this set, by

$$\varphi_{i/n}(x + n\mathbb{Z}) = ix/n + \mathbb{Z}$$

In fact, we see that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Q}/\mathbb{Z})$ is an abelian group isomorphic to $\mathbb{Z}/n$, with

$$\varphi_{1/n}(x + n\mathbb{Z}) = x/n + \mathbb{Z}$$

as a generator. ///

[27.5] Compute $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$.

We claim that this tensor product is isomorphic to $\mathbb{Q}$, via the $\mathbb{Z}$-linear map β induced from the $\mathbb{Z}$-bilinar map $B : \mathbb{Q} \times \mathbb{Q} \longrightarrow \mathbb{Q}$ given by

$$B : x \times y \longrightarrow xy$$

First, observe that the monomials $x \otimes 1$ generate the tensor product. Indeed, given $a/b \in \mathbb{Q}$ (with a, b integers, $b \neq 0$) we have

$$x \otimes \frac{a}{b} = (\frac{x}{b} \cdot b) \otimes \frac{a}{b} = \frac{x}{b} \otimes (b \cdot \frac{a}{b}) = \frac{x}{b} \otimes a = \frac{x}{b} \otimes a \cdot 1 = (a \cdot \frac{x}{b}) \otimes 1 = \frac{ax}{b} \otimes 1$$

proving the claim. Further, any finite $\mathbb{Z}$-linear combination of such elements can be rewritten as a single one: letting $n_i \in \mathbb{Z}$ and $x_i \in \mathbb{Q}$, we have

$$\sum_i n_i \cdot (x_i \otimes 1) = (\sum_i n_i x_i) \otimes 1$$

This gives an outer bound for the size of the tensor product. Now we need an inner bound, to know that there is no *further* collapsing in the tensor product.

From the defining property of the tensor product there *exists* a (unique) $\mathbb{Z}$-linear map from the tensor product to $\mathbb{Q}$, through which B factors. We have $B(x, 1) = x$, so the induced $\mathbb{Z}$-linear map β is a bijection on $\{x \otimes 1 : x \in \mathbb{Q}\}$, so it is an isomorphism. ///

[27.6] Compute $(\mathbb{Q}/\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$.

We claim that the tensor product is 0. It suffices to show that every $m \otimes n$ is 0, since these monomials generate the tensor product. Given $x \in \mathbb{Q}/\mathbb{Z}$, let $0 < n \in \mathbb{Z}$ such that $nx = 0$. For any $y \in \mathbb{Q}$,

$$x \otimes y = x \otimes (n \cdot \frac{y}{n}) = (nx) \otimes \frac{y}{n} = 0 \otimes \frac{y}{n} = 0$$

as claimed. ///

[27.7] Compute $(\mathbb{Q}/\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})$.

We claim that the tensor product is 0. It suffices to show that every $m \otimes n$ is 0, since these monomials generate the tensor product. Given $x \in \mathbb{Q}/\mathbb{Z}$, let $0 < n \in \mathbb{Z}$ such that $nx = 0$. For any $y \in \mathbb{Q}/\mathbb{Z}$,

$$x \otimes y = x \otimes (n \cdot \frac{y}{n}) = (nx) \otimes \frac{y}{n} = 0 \otimes \frac{y}{n} = 0$$

as claimed. Note that we do *not* claim that $\mathbb{Q}/Z$ is a $\mathbb{Q}$-module (which it is not), but only that for given $y \in \mathbb{Q}/\mathbb{Z}$ there is another element $z \in \mathbb{Q}/\mathbb{Z}$ such that $nz = y$. That is, $\mathbb{Q}/Z$ is a **divisible** $\mathbb{Z}$-module.
///

[27.8] Prove that for a subring R of a commutative ring S, with $1_R = 1_S$, polynomial rings $R[x]$ behave well with respect to tensor products, namely that (as rings)

$$R[x] \otimes_R S \approx S[x]$$

Given an R-algebra homomorphism $\varphi : R \longrightarrow A$ and $a \in A$, let $\Phi : R[x] \longrightarrow A$ be the unique R-algebra homomorphism $R[x] \longrightarrow A$ which is φ on R and such that $\varphi(x) = a$. In particular, this works for A an S-algebra and φ the restriction to R of an S-algebra homomorphism $\varphi : S \longrightarrow A$. By the defining property of the tensor product, the bilinear map $B : R[x] \times S \longrightarrow A$ given by

$$B(P(x) \times s) = s \cdot \Phi(P(x))$$

gives a unique R-module map $\beta : R[x] \otimes_R S \longrightarrow A$. Thus, the tensor product has most of the properties necessary for it to be the free S-algebra on one generator $x \otimes 1$.

[6.0.1] **Remark:** However, we might be concerned about verification that each such β is an S-algebra map, rather than just an R-module map. We can certainly write an expression that appears to describe the multiplication, by

$$(P(x) \otimes s) \cdot (Q(x) \otimes t) = P(x)Q(x) \otimes st$$

for polynomials P, Q and $s, t \in S$. *If* it is *well-defined*, then it is visibly associative, distributive, etc., as required.

[6.0.2] **Remark:** The S-module structure itself is more straightforward: for any R-module M the tensor product $M \otimes_R S$ has a natural S-module structure given by

$$s \cdot (m \otimes t) = m \otimes st$$

for $s, t \in S$ and $m \in M$. But one could object that this structure is chosen at random. To argue that this is a *good* way to convert M into an S-module, we claim that for any other S-module N we have a natural isomorphism of abelian groups

$$\mathrm{Hom}_S(M \otimes_R S, N) \approx \mathrm{Hom}_R(M, N)$$

(where on the right-hand side we simply *forget* that N had more structure than that of R-module). The map is given by

$$\Phi \longrightarrow \varphi_\Phi \quad \text{where} \quad \varphi_\Phi(m) = \Phi(m \otimes 1)$$

and has inverse

$$\Phi_\varphi \longleftarrow \varphi \quad \text{where} \quad \Phi_\varphi(m \otimes s) = s \cdot \varphi(m)$$

One might further carefully verify that these two maps are inverses.

[6.0.3] **Remark:** The definition of the tensor product does give an $\mathbb{R}$-linear map

$$\beta : R[x] \otimes_R S \longrightarrow S[x]$$

associated to the R-bilinear $B : R[x] \times S \longrightarrow S[x]$ by

$$B(P(x) \otimes s) = s \cdot P(x)$$

for $P(x) \in R[x]$ and $s \in S$. But it does not seem trivial to prove that this gives an isomorphism. Instead, it may be better to use the universal mapping property of a free algebra. In any case, there would still remain the issue of proving that the induced maps are S-algebra maps.

[27.9] Let K be a field extension of a field k. Let $f(x) \in k[x]$. Show that

$$k[x]/f \otimes_k K \approx K[x]/f$$

where the indicated quotients are by the ideals generated by f in $k[x]$ and $K[x]$, respectively.

Upon reflection, one should realize that we want to prove isomorphism as $K[x]$-modules. Thus, we implicitly use the facts that $k[x]/f$ is a $k[x]$-module, that $k[x] \otimes_k K \approx K[x]$ as K-algebras, and that $M \otimes_k K$ gives a $k[x]$-module M a $K[x]$-module structure by

$$(\sum_i s_i x^i) \cdot (m \otimes 1) = \sum_i (x^i \cdot m) \otimes s_i$$

The map

$$k[x] \otimes_k K \approx_{\text{ring}} K[x] \longrightarrow K[x]/f$$

has kernel (in $K[x]$) exactly of multiples $Q(x) \cdot f(x)$ of $f(x)$ by polynomials $Q(x) = \sum_i s_i x^i$ in $K[x]$. The inverse image of such a polynomial via the isomorphism is

$$\sum_i x^i f(x) \otimes s_i$$

Let I be the ideal generated in $k[x]$ by f, and $\tilde{I}$ the ideal generated by f in $K[x]$. The k-bilinear map

$$k[x]/f \times K \longrightarrow K[x]/f$$

by

$$B : (P(x) + I) \times s \longrightarrow s \cdot P(x) + \tilde{I}$$

gives *a* map $\beta : k[x]/f \otimes_k K \longrightarrow K[x]/f$. The map β is *surjective*, since

$$\beta(\sum_i (x^i + I) \otimes s_i) = \sum_i s_i x^i + \tilde{I}$$

hits every polynomial $\sum_i s_i x^i$ mod $\tilde{I}$. On the other hand, if

$$\beta(\sum_i (x^i + I) \otimes s_i) \in \tilde{I}$$

then $\sum_i s_i x^i = F(x) \cdot f(x)$ for some $F(x) \in K[x]$. Let $F(x) = \sum_j t_j x^j$. With $f(x) = \sum_\ell c_\ell x^\ell$, we have

$$s_i = \sum_{j+\ell=i} t_j c_\ell$$

Then, using k-linearity,

$$\sum_i (x^i + I) \otimes s_i = \sum_i \left(x^i + I \otimes (\sum_{j+\ell=i} t_j c_\ell) \right) = \sum_{j,\ell} \left(x^{j+\ell} + I \otimes t_j c_\ell \right)$$

$$= \sum_{j,\ell} \left(c_\ell x^{j+\ell} + I \otimes t_j \right) = \sum_j (\sum_\ell c_\ell x^{j+\ell} + I) \otimes t_j = \sum_j (f(x) x^j + I) \otimes t_j = \sum_j 0 = 0$$

So the map is a bijection, so is an isomorphism. ///

[27.10] Let K be a field extension of a field k. Let V be a finite-dimensional k-vectorspace. Show that $V \otimes_k K$ is a good definition of the **extension of scalars** of V from k to K, in the sense that for any K-vectorspace W

$$\text{Hom}_K(V \otimes_k K, W) \approx \text{Hom}_k(V, W)$$

where in $\mathrm{Hom}_k(V, W)$ we *forget* that W was a K-vectorspace, and only think of it as a k-vectorspace.

This is a special case of a general phenomenon regarding *extension of scalars*. For any k-vectorspace V the tensor product $V \otimes_k K$ has a natural K-module structure given by

$$s \cdot (v \otimes t) = v \otimes st$$

for $s, t \in K$ and $v \in V$. To argue that this is a *good* way to convert k-vectorspaces V into K-vectorspaces, claim that for any other K-module W have a natural isomorphism of abelian groups

$$\mathrm{Hom}_K(V \otimes_k K, W) \approx \mathrm{Hom}_k(V, W)$$

On the right-hand side we *forget* that W had more structure than that of k-vectorspace. The map is

$$\Phi \longrightarrow \varphi_\Phi \quad \text{where} \quad \varphi_\Phi(v) = \Phi(v \otimes 1)$$

and has inverse

$$\Phi_\varphi \longleftarrow \varphi \quad \text{where} \quad \Phi_\varphi(v \otimes s) = s \cdot \varphi(v)$$

To verify that these are mutual inverses, compute

$$\varphi_{\Phi_\varphi}(v) = \Phi_\varphi(v \otimes 1) = 1 \cdot \varphi(v) = \varphi(v)$$

and

$$\Phi_{\varphi_\Phi}(v \otimes 1) = 1 \cdot \varphi_\Phi(v) = \Phi(v \otimes 1)$$

which proves that the maps are inverses. ///

[6.0.4] **Remark:** In fact, the two spaces of homomorphisms in the isomorphism can be given natural structures of K-vectorspaces, and the isomorphism just constructed can be verified to respect this additional structure. The K-vectorspace structure on the left is clear, namely

$$(s \cdot \Phi)(m \otimes t) = \Phi(m \otimes st) = s \cdot \Phi(m \otimes t)$$

The structure on the right is

$$(s \cdot \varphi)(m) = s \cdot \varphi(m)$$

The latter has only the one presentation, since only W is a K-vectorspace.

[27.11] Let M and N be free R-modules, where R is a commutative ring with identity. Prove that $M \otimes_R N$ is free and

$$\mathrm{rank}\, M \otimes_R N = \mathrm{rank}\, M \cdot \mathrm{rank}\, N$$

Let M and N be free on generators $i : X \longrightarrow M$ and $j : Y \longrightarrow N$. We claim that $M \otimes_R N$ is free on a set map

$$\ell : X \times Y \longrightarrow M \otimes_R N$$

To verify this, let $\varphi : X \times Y \longrightarrow Z$ be a set map. For each fixed $y \in Y$, the map $x \longrightarrow \varphi(x, y)$ factors through a unique R-module map $B_y : M \longrightarrow Z$. For each $m \in M$, the map $y \longrightarrow B_y(m)$ gives rise to a unique R-linear map $n \longrightarrow B(m, n)$ such that

$$B(m, j(y)) = B_y(m)$$

The linearity in the second argument assures that we still have the linearity in the first, since for $n = \sum_t r_t\, j(y_t)$ we have

$$B(m, n) = B(m, \sum_t r_t j(y_t)) = \sum_t r_t B_{y_t}(m)$$

which is a linear combination of linear functions. Thus, there is a unique map to Z induced on the tensor product, showing that the tensor product with set map $i \times j : X \times Y \longrightarrow M \otimes_R N$ is free. ///

[27.12] Let M be a free R-module of rank r, where R is a commutative ring with identity. Let S be a commutative ring with identity containing R, such that $1_R = 1_S$. Prove that as an S module $M \otimes_R S$ is free of rank r.

We prove a bit more. First, instead of simply an *inclusion* $R \subset S$, we can consider any ring homomorphism $\psi : R \longrightarrow S$ such that $\psi(1_R) = 1_S$.

Also, we can consider arbitrary sets of generators, and give more details. Let M be free on generators $i : X \longrightarrow M$, where X is a set. Let $\tau : M \times S \longrightarrow M \otimes_R S$ be the canonical map. We claim that $M \otimes_R S$ is free on $j : X \longrightarrow M \otimes_R S$ defined by

$$j(x) = \tau(i(x) \times 1_S)$$

Given an S-module N, we can be a little forgetful and consider N as an R-module via ψ, by $r \cdot n = \psi(r)n$. Then, given a set map $\varphi : X \longrightarrow N$, since M is free, there is a unique R-module map $\Phi : M \longrightarrow N$ such that $\varphi = \Phi \circ i$. That is, the diagram

$$\begin{tikzcd} M \arrow[dr, dashed, "\Phi"] & \\ X \arrow[u, "i"] \arrow[r, "\varphi"] & N \end{tikzcd}$$

commutes. Then the map

$$\psi : M \times S \longrightarrow N$$

by

$$\psi(m \times s) = s \cdot \Phi(m)$$

induces (by the defining property of $M \otimes_R S$) a unique $\Psi : M \otimes_R S \longrightarrow N$ making a commutative diagram

$$\begin{tikzcd} M \otimes_R S \arrow[dddr, dashed, bend left, "\Psi"] & \\ M \times S \arrow[u, "\tau"] \arrow[ddr, dashed, "\psi"] & \\ X \times \{1_S\} \arrow[u, "i \times inc"] & \\ X \arrow[u, "t"] \arrow[r, "\varphi"] & N \end{tikzcd}$$

where *inc* is the inclusion map $\{1_S\} \longrightarrow S$, and where $t : X \longrightarrow X \times \{1_S\}$ by $x \longrightarrow x \times 1_S$. Thus, $M \otimes_R S$ is free on the composite $j : X \longrightarrow M \otimes_R S$ defined to be the composite of the vertical maps in that last diagram. This argument does not depend upon finiteness of the generating set. ///

[27.13] For finite-dimensional vectorspaces V, W over a field k, prove that there is a natural isomorphism

$$(V \otimes_k W)^* \approx V^* \otimes W^*$$

where $X^* = \mathrm{Hom}_k(X, k)$ for a k-vectorspace X.

For finite-dimensional V and W, since $V \otimes_k W$ is free on the cartesian product of the generators for V and W, the dimensions of the two sides match. We make an isomorphism from right to left. Create a bilinear map

$$V^* \times W^* \longrightarrow (V \otimes_k W)^*$$

as follows. Given $\lambda \in V^*$ and $\mu \in W^*$, as usual make $\Lambda_{\lambda,\mu} \in (V \otimes_k W)^*$ from the bilinear map

$$B_{\lambda,\mu} : V \times W \longrightarrow k$$

defined by

$$B_{\lambda,\mu}(v, w) = \lambda(v) \cdot \mu(w)$$

This induces a unique functional $\Lambda_{\lambda,\mu}$ on the tensor product. This induces a unique linear map

$$V^* \otimes W^* \longrightarrow (V \otimes_k W)^*$$

as desired.

Since everything is finite-dimensional, bijectivity will follow from injectivity. Let $e_1, \ldots, e_m$ be a basis for V, $f_1, \ldots, f_n$ a basis for W, and $\lambda_1, \ldots, \lambda_m$ and $\mu_1, \ldots, \mu_n$ corresponding dual bases. We have shown that a basis of a tensor product of free modules is free on the cartesian product of the generators. Suppose that $\sum_{ij} c_{ij} \lambda_i \otimes \mu_j$ gives the 0 functional on $V \otimes W$, for some scalars c_{ij}. Then, for every pair of indices s, t, the function is 0 on $e_s \otimes f_t$. That is,

$$0 = \sum_{ij} c_{ij} \lambda_i(e_s)\, \lambda_j(f_t) = c_{st}$$

Thus, all constants c_{ij} are 0, proving that the map is injective. Then a dimension count proves the isomorphism. ///

[27.14] For a finite-dimensional k-vectorspace V, prove that the bilinear map

$$B : V \times V^* \longrightarrow \mathrm{End}_k(V)$$

by

$$B(v \times \lambda)(x) = \lambda(x) \cdot v$$

gives an isomorphism $V \otimes_k V^* \longrightarrow \mathrm{End}_k(V)$. Further, show that the composition of endormorphisms is the same as the map induced from the map on

$$(V \otimes V^*) \times (V \otimes V^*) \longrightarrow V \otimes V^*$$

given by

$$(v \otimes \lambda) \times (w \otimes \mu) \longrightarrow \lambda(w) v \otimes \mu$$

The bilinear map $v \times \lambda \longrightarrow T_{v,\lambda}$ given by

$$T_{v,\lambda}(w) = \lambda(w) \cdot v$$

induces a *unique* linear map $j : V \otimes V^* \longrightarrow \mathrm{End}_k(V)$.

To prove that j is injective, we may use the fact that a basis of a tensor product of free modules is free on the cartesian product of the generators. Thus, let $e_1, \ldots, e_n$ be a basis for V, and $\lambda_1, \ldots, \lambda_n$ a dual basis for V^*. Suppose that

$$\sum_{i,j=1}^{n} c_{ij}\, e_i \otimes \lambda_j \longrightarrow 0 \mathrm{End}_k(V)$$

That is, for every e_ℓ,

$$\sum_{ij} c_{ij} \lambda_j(e_\ell) e_i = 0 \in V$$

This is

$$\sum_i c_{ij} e_i = 0 \quad \text{(for all } j\text{)}$$

Since the e_is are linearly independent, all the c_{ij}s are 0. Thus, the map j is injective. Then counting k-dimensions shows that this j is a k-linear isomorphism.

Composition of endomorphisms is a bilinear map

$$\mathrm{End}_k(V) \times \mathrm{End}_k(V) \xrightarrow{\circ} \mathrm{End}_k(V)$$

by

$$S \times T \longrightarrow S \circ T$$

Denote by

$$c : (v \otimes \lambda) \times (w \otimes \mu) \longrightarrow \lambda(w) v \otimes \mu$$

the allegedly corresonding map on the tensor products. The induced map on $(V \otimes V^*) \otimes (V \otimes V^*)$ is an example of a **contraction map** on tensors. We want to show that the diagram

$$\begin{tikzcd}
\mathrm{End}_k(V) \times \mathrm{End}_k(V) \arrow[r, "\circ"] & \mathrm{End}_k(V) \\
(V \otimes_k V^*) \times (V \otimes_k V^*) \arrow[u, "j \times j"] \arrow[r, "c"] & V \otimes_k V^* \arrow[u, "j"]
\end{tikzcd}$$

commutes. It suffices to check this starting with $(v \otimes \lambda) \times (w \otimes \mu)$ in the lower left corner. Let $x \in V$. Going up, then to the right, we obtain the endomorphism which maps x to

$$j(v \otimes \lambda) \circ j(w \otimes \mu)\ (x) = j(v \otimes \lambda)(j(w \otimes \mu)(x)) = j(v \otimes \lambda)(\mu(x)\, w)$$

$$= \mu(x)\, j(v \otimes \lambda)(w) = \mu(x)\, \lambda(w)\, v$$

Going the other way around, to the right then up, we obtain the endomorphism which maps x to

$$j(\, c((v \otimes \lambda) \times (w \otimes \mu)))\ (x) = j(\, \lambda(w)(v \otimes \mu)\,)\ (x) = \lambda(w)\, \mu(x)\, v$$

These two outcomes are the same. ///

[27.15] Under the isomorphism of the previous problem, show that the linear map

$$\mathrm{tr} : \mathrm{End}_k(V) \longrightarrow k$$

is the linear map

$$V \otimes V^* \longrightarrow k$$

induced by the bilinear map $v \times \lambda \longrightarrow \lambda(v)$.

Note that the induced map

$$V \otimes_k V^* \longrightarrow k \quad \text{by} \quad v \otimes \lambda \longrightarrow \lambda(v)$$

is another **contraction map** on tensors. Part of the issue is to compare the coordinate-bound trace with the induced (contraction) map $t(v \otimes \lambda) = \lambda(v)$ determined uniquely from the bilinear map $v \times \lambda \longrightarrow \lambda(v)$. To this end, let $e_1, \ldots, e_n$ be a basis for V, with dual basis $\lambda_1, \ldots, \lambda_n$. The corresponding matrix coefficients $T_{ij} \in k$ of a k-linear endomorphism T of V are

$$T_{ij} = \lambda_i(Te_j)$$

(Always there is the worry about interchange of the indices.) Thus, in these coordinates,

$$\mathrm{tr}\, T = \sum_i \lambda_i(Te_i)$$

Let $T = j(e_s \otimes \lambda_t)$. Then, since $\lambda_t(e_i) = 0$ unless $i = t$,

$$\operatorname{tr} T = \sum_i \lambda_i(Te_i) = \sum_i \lambda_i(j(e_s \otimes \lambda_t)e_i) = \sum_i \lambda_i(\lambda_t(e_i) \cdot e_s) = \lambda_t(\lambda_t(e_t) \cdot e_s) = \begin{cases} 1 & (s = t) \\ 0 & (s \neq t) \end{cases}$$

On the other hand,

$$t(e_s \otimes \lambda_t) = \lambda_t(e_s) = \begin{cases} 1 & (s = t) \\ 0 & (s \neq t) \end{cases}$$

Thus, these two k-linear functionals agree on the monomials, which span, they are equal. ///

[27.16] Prove that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ for two endomorphisms of a finite-dimensional vector space V over a field k, with trace defined as just above.

Since the maps

$$\mathrm{End}_k(V) \times \mathrm{End}_k(V) \longrightarrow k$$

by

$$A \times B \longrightarrow \operatorname{tr}(AB) \quad \text{and/or} \quad A \times B \longrightarrow \operatorname{tr}(BA)$$

are bilinear, it suffices to prove the equality on (images of) monomials $v \otimes \lambda$, since these span the endomophisms over k. Previous examples have converted the issue to one concerning $V_k^{\otimes} V^*$. (We have already shown that the isomorphism $V \otimes_k V^* \approx \mathrm{End}_k(V)$ is converts a *contraction* map on tensors to composition of endomorphisms, and that the trace on tensors defined as another contraction corresponds to the trace of matrices.) Let tr now denote the contraction-map trace on tensors, and (temporarily) write

$$(v \otimes \lambda) \circ (w \otimes \mu) = \lambda(w)\, v \otimes \mu$$

for the contraction-map composition of endomorphisms. Thus, we must show that

$$\operatorname{tr}\ (v \otimes \lambda) \circ (w \otimes \mu) = \operatorname{tr}\ (w \otimes \mu) \circ (v \otimes \lambda)$$

The left-hand side is

$$\operatorname{tr}\ (v \otimes \lambda) \circ (w \otimes \mu) = \operatorname{tr}(\,\lambda(w)\, v \otimes \mu) = \lambda(w) \operatorname{tr}(v \otimes \mu) = \lambda(w)\, \mu(v)$$

The right-hand side is

$$\operatorname{tr}\ (w \otimes \mu) \circ (v \otimes \lambda) = \operatorname{tr}(\,\mu(v)\, w \otimes \lambda) = \mu(v) \operatorname{tr}(w \otimes \lambda) = \mu(v)\, \lambda(w)$$

These elements of k are the same. ///

[27.17] Prove that tensor products are *associative*, in the sense that, for R-modules A, B, C, we have a *natural isomorphism*

$$A \otimes_R (B \otimes_R C) \approx (A \otimes_R B) \otimes_R C$$

In particular, *do* prove the *naturality*, at least the one-third part of it which asserts that, for every R-module homomorphism $f : A \longrightarrow A'$, the diagram

$$\begin{array}{ccc} A \otimes_R (B \otimes_R C) & \xrightarrow{\approx} & (A \otimes_R B) \otimes_R C \\ \Big\downarrow {\scriptstyle f \otimes (1_B \otimes 1_C)} & & \Big\downarrow {\scriptstyle (f \otimes 1_B) \otimes 1_C} \\ A' \otimes_R (B \otimes_R C) & \xrightarrow{\approx} & (A' \otimes_R B) \otimes_R C \end{array}$$

commutes, where the two horizontal isomorphisms are those determined in the first part of the problem. (One might also consider maps $g : B \longrightarrow B'$ and $h : C \longrightarrow C'$, but these behave similarly, so there's no real compulsion to worry about them, apart from awareness of the issue.)

Since all tensor products are over R, we drop the subscript, to lighten the notation. As usual, to make a (linear) map *from* a tensor product $M \otimes N$, we induce uniquely from a bilinear map on $M \times N$. We have done this enough times that we will suppress this part now.

The thing that is slightly less trivial is construction of maps *to* tensor products $M \otimes N$. These are always obtained by composition with the canonical bilinear map

$$M \times N \longrightarrow M \otimes N$$

Important at present is that we can create n-fold tensor products, as well. Thus, we prove the indicated isomorphism by proving that both the indicated iterated tensor products are (naturally) isomorphic to the un-parenthesis'd tensor product $A \otimes B \otimes C$, with canonical map $\tau : A \times B \times C \longrightarrow A \otimes B \otimes C$, such that for every trilinear map $\varphi : A \times B \times C \longrightarrow X$ there is a unique linear $\Phi : A \otimes B \otimes C \longrightarrow X$ such that

$$\begin{tikzcd}
A \otimes B \otimes C \arrow[dr, dashed, "\Phi"] & \\
A \times B \times C \arrow[u, "\tau"] \arrow[r, "\varphi"] & X
\end{tikzcd}$$

The set map

$$A \times B \times C \approx (A \times B) \times C \longrightarrow (A \otimes B) \otimes C$$

by

$$a \times b \times c \longrightarrow (a \times b) \times c \longrightarrow (a \otimes b) \otimes c$$

is linear in each single argument (for fixed values of the others). Thus, we are assured that there is a unique induced linear map

$$A \otimes B \otimes C \longrightarrow (A \otimes B) \otimes C$$

such that

$$\begin{tikzcd}
A \otimes B \otimes C \arrow[dr, dashed, "i"] & \\
A \times B \times C \arrow[u] \arrow[r] & (A \otimes B) \otimes C
\end{tikzcd}$$

commutes.

Similarly, from the set map

$$(A \times B) \times C \approx A \times B \times C \longrightarrow A \otimes B \otimes C$$

by

$$(a \times b) \times c \longrightarrow a \times b \times c \longrightarrow a \otimes b \otimes c$$

is linear in each single argument (for fixed values of the others). Thus, we are assured that there is a unique induced linear map

$$(A \otimes B) \otimes C \longrightarrow A \otimes B \otimes C$$

such that

$$\begin{tikzcd}
(A \otimes B) \otimes C \arrow[dr, dashed, "j"] & \\
(A \times B) \times C \arrow[u] \arrow[r] & A \otimes B \otimes C
\end{tikzcd}$$

commutes.

Then $j \circ i$ is a map of $A \otimes B \otimes C$ to itself compatible with the canonical map $A \times B \times C \longrightarrow A \otimes B \otimes C$. By uniqueness, $j \circ i$ is the identity on $A \otimes B \otimes C$. Similarly (just very slightly more complicatedly), $i \circ j$ must be the identity on the iterated tensor product. Thus, these two maps are mutual inverses.

To prove naturality in one of the arguments A, B, C, consider $f : C \longrightarrow C'$. Let j_{ABC} be the isomorphism for a fixed triple A, B, C, as above. The diagram of maps of cartesian products (of sets, at least)

$$\begin{tikzcd} (A \times B) \times C \arrow[r, "j_{ABC}"] \arrow[d, "(1_A \times 1_B) \times f"] & A \times B \times C \arrow[d, "1_A \times 1_B \times f"] \\ (A \times B) \times C \arrow[r, "j"] & A \times B \times C \end{tikzcd}$$

does commute: going down, then right, is

$$j_{ABC'}\left((1_A \times 1_B) \times f\right)((a \times b) \times c)) = j_{ABC'}\left((a \times b) \times f(c)\right) = a \times b \times f(c)$$

Going right, then down, gives

$$(1_A \times 1_B \times f)\left(j_{ABC}((a \times b) \times c)\right) = (1_A \times 1_B \times f)\left(a \times b \times c\right)) = a \times b \times f(c)$$

These are the same. ///

Exercises

27.[6.0.1] Let I and J be two ideals in a PID R. Determine

$$R/I \otimes_R R/J$$

27.[6.0.2] For an R-module M and an ideal I in R, show that

$$M/I \cdot M \approx M \otimes_R R/I$$

27.[6.0.3] Let R be a commutative ring with unit, and S a commutative R algebra. Given an R-bilinear map $B : V \times W \longrightarrow R$, give a natural S-blinear extension of B to the S-linear extensions $S \otimes_R V$ and $S \otimes_R W$.

27.[6.0.4] A **multiplicative subset** S of a commutative ring R with unit is a subset of R closed under multiplication. The **localization** $S^{-1}R$ of R at S is the collection of ordered pairs (r, s) with $r \in R$ and $s \in S$, modulo the equivalence relation that $(r, s) \sim (r', s')$ if and only if there is $s'' \in S$ such that

$$s'' \cdot (rs' - r's) = 0$$

Let P be a prime ideal in R. Show that $S^{-1}P$ is a prime ideal in $S^{-1}R$.

27.[6.0.5] In the situation of the previous exercise, show that the field of fractions of $(S^{-1}R)/(S^{-1}P)$ is naturally isomorphic to the field of fractions of R/P.

27.[6.0.6] In the situation of the previous two exercises, for an R-module M, define a reasonable notion of $S^{-1}M$.

27.[6.0.7] In the situation of the previous three exercises, for an R-module M, show that

$$S^{-1}M \approx M \otimes_R S^{-1}R$$

27.[6.0.8] Identify the commutative $\mathbb{Q}$-algebra $\mathbb{Q}(\sqrt{2}) \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{2})$ as a sum of fields.

27.[6.0.9] Identify the commutative $\mathbb{Q}$-algebra $\mathbb{Q}(\sqrt[3]{2}) \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt[3]{2})$ as a sum of fields.

27.[6.0.10] Let ζ be a primitie 5^{th} root of unity. Identify the commutative $\mathbb{Q}$-algebra $\mathbb{Q}(\sqrt{5}) \otimes_{\mathbb{Q}} \mathbb{Q}(\zeta)$ as a sum of fields.

27.[6.0.11] Let $\mathfrak{H}$ be the Hamiltonian quaternions. Identify $\mathfrak{H} \otimes_{\mathbb{R}} \mathbb{C}$ in familiar terms.

27.[6.0.12] Let $\mathfrak{H}$ be the Hamiltonian quaternions. Identify $\mathfrak{H} \otimes_{\mathbb{R}} \mathfrak{H}$ in familiar terms.