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Altenating grouops, commutator subgroups

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- Commutator subgroups and abelianization
- Alternating groups
- Linear groups

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1. *Commutator subgroups and abelianization*

The *commutator* of two elements x, y of a group G is [1]

$$[x, y] \;=\; xyx^{-1}y^{-1}$$

The *commutator subgroup* $[G, G]$ of G is the smallest subgroup of G containing all commutators. [2] It is immediate that G is abelian if and only if all commutators are 1. At the other extreme, it can happen that $G = [G, G]$, in which case G is called *perfect*.

[1.1] Proposition: Let $f : G \to A$ be a group homomorphism to an abelian group. Then $\ker f$ contains the commutator subgroup $[G, G]$. Conversely, the commutator subgroup is *normal* and the quotient $G/[G, G]$ is abelian.

Proof: The main point of this is that, for a homomorphism $f : G \to A$ to an abelian group A,

$$f(xyx^{-1}y^{-1}) \;=\; f(x)\,f(y)\,f(x^{-1})\,f(y^{-1}) \;=\; f(x)f(x^{-1})\;f(y)f(y^{-1}) \;=\; 0$$

so commutators are certainly in the kernel of f. For the converse, observe first that

$$z\cdot[x, y]\cdot z^{-1} \;=\; z\cdot(xyx^{-1}y^{-1})\cdot z^{-1} \;=\; (zxz^{-1})\,(zyz^{-1})\,(zxz^{-1})^{-1}\,(zyz^{-1})^{-1} \;=\; [zxz^{-1}, zyz^{-1}]$$

That is, the set of commutators is stable under conjugation. Thus, for a subgroup H containing all commutators, zHz^{-1} also contains all commutators. Then

$$z[G, G]z^{-1} \;=\; z\left(\bigcap_{H\ni\text{commutators}} H\right) \;=\; \bigcap_{H\ni\text{commutators}} zHz^{-1} \;=\; \bigcap_{H\ni\text{commutators}} H \;=\; [G, G]$$

proving the normality of the commutator subgroup. [3] Let $q : G \to [G, G]$ be the quotient map. To show that the quotient is abelian, consider commutators in the quotient

$$q(x)q(y)q(x)^{-1}q(y)^{-1} = q(xyx^{-1}y^{-1}) = 1$$

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[1] Beware that in other circumstances the same notation has different meanings. In a ring it may be that $[x, y] = xy - yx$. And in a *Lie algebra* (an important and useful type of non-associative algebra) the ring operation itself is written as $[x, y]$ rather than multiplication, both to avoid suggesting associativity, and because it is in fact descended from the group commutator.

[2] As usual, this language means that the commutator subgroup is the *intersection* of all subgroups containing all commutators. The intersection of any family of subgroups is a subgroup.

[3] Note that we did *not* need to refer to explicit algebraic expressions involving commutators of elements.

Since these are all trivial, the quotient is abelian. ///

[1.2] Corollary: If a finite group G is *simple*[4] then it is equal to its own commutator subgroup. ///

[1.3] Corollary: If a non-trivial finite group G is *solvable*[5] then its commutator subgroup is a *proper* subgroup.

Proof: If G is not already cyclic, then it has a normal subgroup G_1 such that G/G_1 is cyclic. In particular, G/G_1 is abelian, so G_1 must contain the commutator subgroup. ///

[1.4] Remark: We could also characterize the *abelianization* $G/[G,G]$ more instrinsically, by saying that it is the *smallest* quotient of G such that every group homomorphism $f : G \to A$ to an abelian group factors through this quotient. More precisely, define *an* abelianization of G to be an abelian group G^{ab} equipped with a homomorphism

$$q : G \longrightarrow G^{\mathrm{ab}}$$

such that for any group homomorphism $f : G \to A$ to an abelian group, there is a unique $g : G^{\mathrm{ab}} \to A$ such that

$$f \;=\; g \circ q : G \xrightarrow{q} G^{\mathrm{ab}} \xrightarrow{g} A$$

As usual when something is defined by such a universal property, we can prove that any two abelianizations (assuming they exist) are uniquely isomorphic, as follows.

First, with $f = q : G \to G^{\mathrm{ab}}$, the uniqueness part of the definition of G^{ab} implies that the identity map 1 on G^{ab} is the only map of G^{ab} to itself compatible with q, that is, such that $1 \circ q = q$. Among other things, this proves that $q : G \to G^{\mathrm{ab}}$ is a surjection.

Next, let $q_i : G \to H_i$ for $i = 1, 2$ be two abelianizations. Then there is a unique $g_1 : H_1 \to H_2$ such that $q_2 = g_1 \circ q_1$, and, symmetrically, there is a unique $g_2 : H_2 \to H_1$ such that $q_1 = g_2 \circ q_2$. Then $g_2 \circ g_1 : H_1 \to H_1$ and $g_1 \circ g_2 : H_2 \to H_2$ are maps of the H_i to themselves and are compatible with $q_i : G \to H_i$. Thus, they are the identity maps on the H_i, so g_1 and g_2 are mutual inverses.

By this point we can be confident that whatever *construction* of an abelianization we choose, the resulting object will be the same. In effect, the proposition above about $G/[G,G]$ proves that this quotient (with the natural map of G to it) is an abelianization.

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2. *Alternating groups*

[2.1] Proposition: For $n \ge 2$, the commutator subgroup $[S_n, S_n]$ of the symmetric group S_n on n things is the alternating group[6] A_n. In particular, all 3-cycles are commutators, and A_n is generated by 3-cycles. (For $n = 2$ this is vacuously true.)

Proof: Certainly commutators are *even* permutations, so $[S_n, S_n] \subset A_n$. For $1 \le i < n$ let s_i be the i^{th}

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[4] That is, it has no proper normal subgroups, and, by convention, is not cyclic of prime order.

[5] As usual, this means that there is a chain of subgroups $G = G_o \supset \ldots \supset G_n$ such that G_{i+1} is normal in G_i, and such that all quotients G_i/G_{i+1} are cyclic.

[6] As usual, the alternating group is the subgroup of S_n consisting of *even* permutations, that is, those expressible as a product of an even number of 2-cycles.

adjacent transposition, that is the 2-cycle interchanging i and $i+1$. For $1 \le i \le n-2$

$$(s_i\, s_{i+1})(j) \;=\; \begin{cases} j & (\text{for } j \neq i, i+1, i+2) \\ i+2 & (\text{for } j = i+1) \\ i & (\text{for } j = i+2) \\ i+1 & (\text{for } j = i) \end{cases}$$

which is a 3-cycle t_i. Thus, we compute a commutator

$$s_i s_{i+1} s_i^{-1} s_{i+1}^{-1} \;=\; s_i s_{i+1} s_i s_{i+1} \;=\; t_i^2 = t_i^{-1}$$

Thus, every 3-cycle on adjacent elements $i, i+1, i+2$ is in the commutator subgroup $[S_n, S_n]$. We now prove that any product $s_i s_j$ of two adjacent transpositions is expressible as a product of these particular 3-cycles t_i. Indeed, for $i < j$, we have collapsing

$$t_i\, t_{i+1}\, t_{i+2} \ldots t_{j-1}\, t_j \;=\; (s_i s_{i+1})\,(s_{i+1} s_{i+2}) \ldots (s_j s_{j+1}) \;=\; s_i\, s_{j+1}$$

Since the adjacent transpositions s_i generate S_n, the products of pairs of adjacent transpositions generate A_n. ///

[2.2] Proposition: For $n \ge 5$, $[A_n, A_n] = A_n$.

Proof: All 3-cycles are in A_n. Then

$$t_1 t_3 t_1^{-1} t_3^{-1} \;=\; s_1 s_2 s_3 s_4 (s_2 s_1) s_4 s_3 \;=\; s_1 s_2 s_3 (s_2 s_1) s_4 s_4 s_3 \;=\; s_1 s_2 s_3 s_2 s_1 s_3$$

using the fact that s_1 and s_2 commute with s_4. This permutation has the effect, traced through its 6 steps for each of $1, 2, 3, 4$,

$$\begin{array}{c} 1 \to 1 \to 2 \to 3 \to 4 \to 4 \to 4 \\ 2 \to 2 \to 1 \to 1 \to 1 \to 1 \to 2 \\ 3 \to 4 \to 4 \to 4 \to 3 \to 2 \to 1 \\ 4 \to 3 \to 3 \to 2 \to 2 \to 3 \to 3 \end{array}$$

That is, the result is the 3-cycle $1 \to 4 \to 3 \to 1$. Once this artifact is discovered, it is clear that a suitable choice of 3-cycles will give any desired 3 cycle as commutator. [7] ///

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3. *Linear groups*

[3.1] Proposition: For a field k with $|k| \ge 4$, and for $n \ge 2$, the group $SL_n(k)$, consisting of n-by-n matrices with entries in k and determinant 1, is its own commutator subgroup.

Proof: The essential point is already visible in $SL_2(k)$. *[... iou ...]* ///

[7] It is a little strange that the extra room $n \ge 5$ is needed to achieve the effect used in this proof.