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Half-exactness of adjoint functors, Yoneda lemma

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Our goal is proof that functors

$$M \to M \otimes X$$

(for example, from $\mathbb{Z}$-modules to $\mathbb{Z}$-modules) are *right exact*. Direct proof is non-trivial. The more pleasant argument introduces **adjoint functors** and proves a simple form of Yoneda's lemma. The argument illustrates **functoriality** of isomorphisms.

To reduce complications and lighten the notation, we treat only $\mathbb{Z}$-modules (that is, abelian groups). In particular, spaces $\mathrm{Hom}(A, B)$ are again abelian groups, as are tensor products $A \otimes B$, so these stay inside the category of $\mathbb{Z}$-modules.

- $M \to \mathrm{Hom}(X, M)$ is left exact
- Adjointness of Hom and $\otimes$
- Yoneda lemma
- Half-exactness of adjoint functors

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1. $M \to \mathrm{Hom}(X, M)$ *is left exact*

The proof is straightforward.

[1.0.1] **Theorem:** The functor $M \to \mathrm{Hom}(X, M)$ is left exact. That is,

$$0 \to A \xrightarrow{i} B \xrightarrow{q} C \to 0 \quad \text{exact} \qquad \Longrightarrow \qquad 0 \to \mathrm{Hom}(X, A) \xrightarrow{i\circ -} \mathrm{Hom}(X, B) \xrightarrow{q\circ -} \mathrm{Hom}(X, C) \quad \text{exact}$$

where the induced maps are by composition with i and with q as indicated. Similarly, for the other Hom functor $M \to \mathrm{Hom}(M, X)$ attached to X,

$$0 \to A \xrightarrow{i} B \xrightarrow{q} C \to 0 \quad \text{exact} \qquad \Longrightarrow \qquad 0 \to \mathrm{Hom}(C, X) \xrightarrow{-\circ q} \mathrm{Hom}(B, X) \xrightarrow{-\circ i} \mathrm{Hom}(C, X) \quad \text{exact}$$

[1.0.2] **Remark:** The Hom functor $M \to \mathrm{Hom}(X, M)$ is *covariant*, in the usual sense that a morphism $f : M \to N$ gives an arrow in the *same* direction

$$\mathrm{Hom}(X, M) \xrightarrow{f\circ -} \mathrm{Hom}(X, N)$$

The other Hom functor $M \to \mathrm{Hom}(M, X)$ is *contravariant*, in the usual sense that a morphism $f : M \to N$ gives an arrow in the *opposite* direction

$$\mathrm{Hom}(N, X) \xrightarrow{-\circ f} \mathrm{Hom}(M, X)$$

Proof: For $f \in \mathrm{Hom}(X, A)$, $i \circ f = 0$ implies $(i \circ f)(x) = 0$ for all $x \in X$, and then $f(x) = 0$ for all x since i is an injection. Thus, $\mathrm{Hom}(X, A) \to \mathrm{Hom}(X, B)$ is an injection, giving exactness at the left joint.

Since $q \circ i = 0$, any $f \in \mathrm{Hom}(X, A)$ is mapped to $0 \in \mathrm{Hom}(X, C)$ by $f \to q \circ i \circ f$. That is, the image of $i \circ -$ is contained in the kernel of $q \circ -$. On the other hand, when $g \in \mathrm{Hom}(X, B)$ is mapped to $q \circ g = 0$ in $\mathrm{Hom}(X, C)$,

$$g(X) \subset \ker q \;=\; \mathrm{Im}\, i$$

Since i is injective, it is an isomorphism to its image, so there is an inverse $i^{-1} : i(A) \to A$. Since $g(X) \subset \mathrm{Im}\, i$ we can define

$$f \;=\; i^{-1} \circ g \in \mathrm{Hom}(X, A)$$

Certainly $i \circ f = g$, so the kernel is contained in the image. This gives exactness at the middle joint, and the left exactness. The exactness of the other Hom is similar. ///

[1.0.3] **Remark:** The functor $M \to \mathrm{Hom}(X, M)$ is *not* right exact. For example, with

$$0 \to \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \to \mathbb{Z}/n \to 0$$

with an integer $n > 1$, with $X = \mathbb{Z}/n$ there is no non-zero map of the torsion abelian group X to the free abelian group $\mathbb{Z}$. Similarly, the (contravariant) functor $M \to \mathrm{Hom}(M, X)$ is not right exact.

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2. *Adjointness of* Hom *and* $\otimes$

Here we introduce **adjoint functors** and give the principal example, adjointness between Hom functors and tensor product functors. The **functoriality** of the isomorphism is explained, and the importance of this functoriality will be illustrated in proving the right exactness of $A \to A \otimes X$.

The adjointness property is related to **Frobenius reciprocity** and **Shapiro's Lemma**.

Let R and L be two functors from the category of $\mathbb{Z}$-modules to itself. These two functors are **mutually adjoint** when, there is a *functorial* isomorphism

$$\mathrm{Hom}(LA, B) \approx \mathrm{Hom}(A, RB) \qquad \text{(for all } A, B)$$

The functor R is a **right adjoint**, and L is a **left adjoint**. *Functoriality* means that, for each pair of morphisms $f : A' \to A$ and $g : B \to B'$ (yes, the maps go from A' to A, but from B to B') we have a commutative diagram[1]

$$\begin{array}{ccccc} & \mathrm{Hom}(LA, B) & \approx & \mathrm{Hom}(A, RB) & \\ g \circ (*) \circ Lf & \downarrow & & \downarrow & Rg \circ (*) \circ f \\ & \mathrm{Hom}(LA', B') & \approx & \mathrm{Hom}(A', RB') & \end{array}$$

That is,

$$g \circ F \circ Lf = Rg \circ F \circ f \qquad \text{(for every } F \in \mathrm{Hom}(LA, B))$$

[2.0.1] **Theorem:** For $\mathbb{Z}$-modules A, X, B we have a *functorial* isomorphism

$$\mathrm{Hom}(A \otimes X, B) \;\approx\; \mathrm{Hom}(A, \mathrm{Hom}(X, B))$$

Proof: Given $\Phi \in \mathrm{Hom}(A \otimes X, B)$, define $\varphi_\Phi \in \mathrm{Hom}(A, \mathrm{Hom}(X, B))$ by

$$\varphi_\Phi(a)(x) = \Phi(a \otimes x)$$

Conversely, given $\varphi \in \mathrm{Hom}(A, \mathrm{Hom}(X, B))$, define $\Phi_\varphi \in \mathrm{Hom}(A \otimes X, B)$ by

$$\Phi_\varphi(a \otimes x) = \varphi(a)(x)$$

[1] Assembling these isomorphisms into larger diagrams is critical in proving the half-exactness results below.

and extending by linearity. Visibly the maps $\Phi \to \varphi_\Phi$ and $\varphi \to \Phi_\varphi$ are mutual inverses.

The *functoriality* of the isomorphism refers to the behavior of the isomorphism when we have $f : A' \to A$ and $g : X \to X'$ and/or $h : B \to B'$. (Yes, the order of the primed and unprimed symbols is opposite.) Thus, the diagram

$$\begin{array}{ccccc} & \mathrm{Hom}(A \otimes X, B) & \approx & \mathrm{Hom}(A, \mathrm{Hom}(X, B)) & \\ \Phi \to g \circ \Phi \circ (f \otimes \mathrm{id}_X) & \downarrow & & \downarrow & \varphi \to (a' \to g(\varphi(f(a'))(x))) \\ & \mathrm{Hom}(A' \otimes X, B') & \approx & \mathrm{Hom}(A', \mathrm{Hom}(X, B')) & \end{array}$$

must commute. This is very easy to check: starting with Φ in the upper left, going down gives $\Phi \circ (f \otimes \mathrm{id}_X)$, and then going to the right gives φ such that

$$\varphi(a')(x) = (\Phi \circ (f \otimes \mathrm{id}_X))(f(a') \otimes x) = \Phi(a \otimes x)$$

Going the other way around the diagram, first we obtain φ such that $\varphi(a)(x) = \Phi(a \otimes x)$. Going down the right side gives φ' such that

$$\varphi'(a')(x) = \varphi(f(a'))(x) = \Phi(f(a') \otimes x)$$

which is the same as the first computation, so we have the functoriality. ///

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3. *Yoneda's lemma*

While proving the right exactness of $A \to A \otimes X$ using results above, the following issues arise. This complement to the left-exactness of $M \to \mathrm{Hom}(X, M)$ is a special case of **Yoneda's Lemma**. [2]

[3.0.1] **Theorem:** We have *sufficient* criteria for exactness:

$$\mathrm{Hom}(X, A) \xrightarrow{f\circ -} \mathrm{Hom}(X, B) \xrightarrow{g\circ -} \mathrm{Hom}(X, C) \quad \text{exact for all } X \qquad \Longrightarrow \qquad A \xrightarrow{f} B \xrightarrow{g} C \quad \text{exact}$$

Also,

$$\mathrm{Hom}(C, X) \xrightarrow{-\circ g} \mathrm{Hom}(B, X) \xrightarrow{-\circ f} \mathrm{Hom}(A, X) \quad \text{exact for all } X \qquad \Longrightarrow \qquad A \xrightarrow{f} B \xrightarrow{g} C \quad \text{exact}$$

[3.0.2] **Remark:** Exactness of $A \to B \to C$ does *not* imply exactness of the Hom diagram for all X. This was visible in proving *left* exactness of $M \to \mathrm{Hom}(M, X)$.

Proof: On one hand, with $X = A$ and $F : X \to A$ the identity, exactness of the Hom sequence implies

$$0 = g \circ f \circ F \;=\; g \circ f$$

so $\mathrm{Im} f \subset \ker g$. On the other hand, with $X = \ker g$ and $F : X \to B$ the inclusion, exactness of the Hom sequence (with $g \circ F = 0$) implies that there is $F' : X \to A$ such that $f \circ F' = F$. Then

$$\ker g = \mathrm{Im} F = \mathrm{Im}(f \circ F') \subset \mathrm{Im} f$$

Putting the two containments together gives $\ker g = \mathrm{Im} f$. This proves the result for the covariant Hom functor.

[2] Such a map $X \to \mathrm{Hom}(X, A)$ of objects, from a category whose sets $\mathrm{Hom}(A, B)$ of maps are abelian groups, to the category of abelian groups, is called a **Yoneda imbedding**.

For the contravariant Hom functor $M \to \mathrm{Hom}(M, X)$, with $X = C$ and $F : C \to X$ the identity, the exactness of the Hom sequence gives

$$0 = F \circ g \circ f = g \circ f$$

Thus, $\mathrm{Im}\, f \subset \ker g$. On the other hand, with $X = B/\mathrm{Im}\, f$ and $F : B \to X$ the quotient map, by exactness of the Hom sequence there is $F' : C \to X$ such that $F' \circ g = F$. Thus, the kernel of g cannot be larger than $\mathrm{Im}\, f$, or $F : B \to B/\mathrm{Im}\, f$ could not factor through it. Thus, we have exactness. ///

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4. *Half-exactness of adjoint functors*

[4.0.1] **Theorem:** Let L, R be adjoint functors on $\mathbb{Z}$-modules, in the sense that there is a *functorial* isomorphism

$$\mathrm{Hom}(LA, B) \approx \mathrm{Hom}(A, RB) \qquad \text{(for every } A, B\text{)}$$

Then L is right half-exact and R is left half-exact. That is, for

$$0 \to A \to B \to C \to 0 \quad \text{exact} \qquad \Longrightarrow \qquad LA \to LB \to LC \to 0 \quad \text{exact}$$

and

$$0 \to A \to B \to C \to 0 \quad \text{exact} \qquad \Longrightarrow \qquad 0 \to RA \to RB \to RC \quad \text{exact}$$

Proof: Left exactness of $M \to \mathrm{Hom}(X, M)$ for any X applies to X replaced by LX, so

$$0 \to \mathrm{Hom}(LX, A) \to \mathrm{Hom}(LX, B) \to \mathrm{Hom}(LX, C) \qquad \text{exact}$$

By adjointness of L and R, and *functoriality* of the adjointness isomorphisms, we have a commutative diagram with exact top row,

$$\begin{array}{ccccccc} 0 & \to & \mathrm{Hom}(LX, A) & \to & \mathrm{Hom}(LX, B) & \to & \mathrm{Hom}(LX, C) \\ & & \approx\downarrow & & \approx\downarrow & & \approx\downarrow \\ 0 & \to & \mathrm{Hom}(X, RA) & \to & \mathrm{Hom}(X, RB) & \to & \mathrm{Hom}(X, RC) \end{array}$$

Then the bottom row is exact, for all X. By Yoneda's lemma,

$$0 \to RA \to RB \to RC \qquad \text{exact}$$

Similarly, for the other Hom functor, for all X we have a commutative diagram with exact top row,

$$\begin{array}{ccccccc} 0 & \to & \mathrm{Hom}(C, RX) & \to & \mathrm{Hom}(B, RX, B) & \to & \mathrm{Hom}(A, RX) \\ & & \approx\downarrow & & \approx\downarrow & & \approx\downarrow \\ 0 & \to & \mathrm{Hom}(LC, X) & \to & \mathrm{Hom}(LB, X) & \to & \mathrm{Hom}(LC, X) \end{array}$$

Then the bottom row is exact, for all X, and by Yoneda

$$LA \to LB \to LC \to 0 \qquad \text{exact}$$

since this second Hom functor $M \to \mathrm{Hom}(M, X)$ is *contravariant*. ///

[4.0.2] **Corollary:** The natural (adjointness) isomorphism $\mathrm{Hom}(A \otimes X, B) \approx \mathrm{Hom}(A, \mathrm{Hom}(X, B))$ yields the left exactness of $M \to \mathrm{Hom}(X, M)$ and the right exactness of $M \to M \otimes X$. ///
