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Outline of basic complex analysis

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Construction of complex numbers as quotient ring of polynomials with real coefficients: $\mathbb{C} = \mathbb{R}[X]/\langle X^2+1\rangle$

Elementary algebra and geometry of complex numbers: multiplication is $(a+bi)(c+di) = (ac-bd)+i(bc+ad)$, *conjugation* $\overline{a+bi} = a - bi$ is the unique $\mathbb{R}$-linear field automorphism of $\mathbb{C}$ other than the identity map.

Absolute value $|a+bi| = \sqrt{a^2+b^2} = \sqrt{(a+bi)(a-bi)} = \sqrt{(a+bi)\overline{(a+bi)}}$. Conjugation preserves multiplication in the sense that $\overline{\alpha\beta} = \overline{\alpha}\cdot\overline{\beta}$ so absolute value preserves multiplication.

Metric: for $\alpha,\beta\in\mathbb{C}$, distance from α to β is $|\alpha-\beta|$. Matches Euclidean distance.

The exponential function $e^z = \sum_{n\ge 0} z^n/n!$, property $e^{z+w} = e^z\cdot e^w$ from binomial theorem $(x+y)^n = \sum_{j=0}^n \binom{n}{j} x^j\, y^{n-j}$, where $\binom{n}{j} = \frac{n!}{j!\,(n-j)!}$

Under multiplication, lengths *multiply*, angles *add*

Euler's identity $e^{i\theta} = \cos\theta + i\sin\theta$, trig functions in terms of exponentials: $\cos z = \frac{e^{iz}+e^{-iz}}{2}$, $\sin z = \frac{e^{iz}-e^{-iz}}{2i}$, trigonometric identities

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Abel's theorem: *real-analytic* functions, that is, functions on open subsets of $\mathbb{R}$ given by convergent power series, are *differentiable*, and the derivative is given by term-wise differentiation of the power series

Complex differentiation: $f'(z) = \lim_{h\to 0}\frac{f(z+h)-f(z)}{h}$ with h *complex*

Abel's theorem for complex power series: convergent power series are *complex*-differentiable

Complex-differentiable functions f *preserve angles* at points z_o with $f'(z_o)\neq 0$, that is, are *conformal*

Examples of conformal mappings

Path integrals $\int_\gamma f = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt$ for $\gamma : [a,b]\to\mathbb{C}$. Independence of parametrization

Winding number of γ about z_o is $\frac{1}{2\pi i}\int_\gamma \frac{dz}{z-z_o}$. Introduction to *homotopy* and *homology*

Cauchy-Goursat theorem: complex-differentiable implies vanishing of path integrals $\int_\gamma f(z)\ dz$ around *triangles* γ

Simplest case of Cauchy formulas: for γ a simple closed curve traced counter-clockwise, with z_o in its interior, $f(z_o) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)\,dz}{z-z_o}$, and $f^{(n)}(z_o) = \frac{n!}{2\pi i}\int_\gamma \frac{f(z)\,dz}{(z-z_o)^{n+1}}$,

Cauchy: complex-differentiable functions f have convergent power series expansions
$f(z) = \sum_{n\ge 0} c_n\,(z-z_o)^n$, where as expected $c_n = \frac{f^{(n)}(z_o)}{n!}$

Liouville's theorem: *bounded* complex-differentiable functions are *constants*. Corollary (sometimes called the *fundamental theorem of algebra*): any complex-coefficiented polynomial of degree n has n zeros in $\mathbb{C}$.

Complex differentiability of f implies Cauchy-Riemann equation $\frac{\partial f}{\partial \overline{z}} = 0$. Separating real and imaginary parts, with $f(x+iy) = u(x,y) + iv(x,y)$, the Cauchy-Riemann equation becomes

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial iy} = -i\frac{\partial f}{\partial y} \qquad \text{or} \qquad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

Holomorphy as synonym for complex-differentiability and for complex-analyticity

Identity principle: If two holomorphic functions on a connected open set agree at a sequence of points having a limit point in that open set, then the two functions are equal *everywhere*. Applications to proving identities.

Logarithms $\log z = \int_1^z \frac{dz}{z}$, multi-valued *argument* function. Failure of $\log(zw) = \log z + \log w$ without additional hypotheses

Argument principle: the number of 0's of f, counting multiplicities, inside a simple closed curve γ is $\frac{1}{2\pi i}\int_\gamma \frac{f'}{f}$.

Isolated singularities, Laurent expansions $f(z) = \sum_{n\in\mathbb{Z}} c_n\,(z-z_o)^n$ for $r' < |z-z_o| < R'$. Formulas for coefficients, for $r' < r < R < R'$, with $\gamma_r = \{z : |z-z_o| = r\}$ and $\gamma_R = \{z : |z-z_o| = R\}$,

$$c_n = \begin{cases} \dfrac{1}{2\pi i}\displaystyle\int_{\gamma_R} \frac{f(\zeta)\,d\zeta}{(\zeta-z_o)^{n+1}} & (\text{for } n \geq 0) \\[2ex] \dfrac{1}{2\pi i}\displaystyle\int_{\gamma_r} \frac{f(\zeta)\,d\zeta}{(\zeta-z_o)^{-n+1}} & (\text{for } n < 0) \end{cases}$$

The *residue* $\mathrm{Res}_{z=z_o} f(z)$ of f at isolated singularity z_o is the -1 Laurent coefficient.

Simple case of *residue theorem*: for simple closed curve γ, summing over (isolated) singularities z_o of f inside γ, $\int_\gamma f(z)\,dz = 2\pi i \sum_{z_o} \mathrm{Res}_{z=z_o} f(z)$.

Evaluation of integrals by residues: examples

Basic ideas about *homotopy* and *homology*, and fancier versions of Cauchy's theorems

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Maximum modulus principle: maximum absolute value occurs on the *boundary*, and is strictly greater than interior points except for *constant* functions.

Rouché's theorem counting zeros of nearby functions: for a simple closed curve γ in an open set U, and f, g holomorphic on U with $|f-g| < |f|$ on γ, then f and g have the same number of zeros inside γ.

Corollaries of Rouché: open mapping theorem, analytic dependence of roots on parameters, ...

More on isolated singularities: poles (finitely-many negative-index Laurent terms) versus *essential* singularities (infinitely-many negative-index Laurent terms). *Meromorphic* functions have only *poles*.

Casorati-Weierstrass theorem: in every neighborhood of an essential singularity of a function, the function comes arbitrarily near every complex value.

Morera's theorem: vanishing of integals along small closed paths implies holomorphy. Indeed, vanishing of integrals along all small *triangles* suffices.

Corollary: uniform pointwise limits of holomorphic functions are holomorphic

Corollary: Schwarz' reflection principle: Let U be a non-empty open set inside the upper half-plane, with the closure of U meeting $\mathbb{R}$ in an interval I. Any function f holomorphic on U and extending continuously to $U \cup I$ extends to a holomorphic function on $U \cup I \cup U^{\mathrm{ref}}$, where U^{ref} is the copy of U *reflected* across the real axis, namely, $U^{\mathrm{ref}} = \{\overline{z} : z \in U\}$, by the formula $f(\overline{z}) = \overline{f(z)}$.

Variant reflection principle: replace the real line with the unit circle, and complex conjugation $z \to \overline{z}$ with $z \to 1/\overline{z}$.

Linear fractional (Möbius) transformations $\begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \to \frac{az+b}{cz+d}$

Schwarz' lemma: For f holomorphic on the open unit disk in $\mathbb{C}$ with $|f(z)| < 1$ on that disk and $f(0) = 0$, then $|f(z)| \le |z|$ for all z in the disk, and $|f'(0)| \le 1$. Further, if $|f(z)| = |z|$ for some z, or if $|f'(0)| = 1$, then $f(z) = c \cdot z$ for some $|c| = 1$.

Automorphisms of the Riemann sphere, of the disk, of the upper half-plane. Hyperbolic 2-space: Poincaré model, Beltrami model.

Riemann mapping theorem

Example: disks with concentric slits

Harmonic functions: mean value theorem, Poisson's integral formula for disks

Harmonic functions in punctured disks

Partial fraction expansions of functions with prescribed poles, such as

$$\frac{\pi^2}{\sin^2 \pi z} \;=\; \sum_{n \in \mathbb{Z}} \frac{1}{(z-n)^2}$$

Weierstrass product expansions of entire functions with given zeros. Example: Euler's factorization

$$\sin \pi z = \pi z \cdot \prod_{n=1}^{\infty} \Big(1 - \frac{z^2}{n^2}\Big)$$

Order and genus of entire functions

Hadamard product expansions of entire functions

Differential equations $u'' + b(z)\,u' + c(z)\,u = 0$, ordinary points, regular singular points, asymptotics

Algebraic functions and Riemann surfaces

Elliptic integrals, elliptic functions (doubly-periodic meromorphic functions). Weierstrass' equation $\wp'(z)^2 = 4\wp(z)^3 - 60g_2\wp(z) - 140g_3$

Elliptic modular functions

Genus of a compact, connected surface is the number of *handles* (!?!?!)

The uniformization theorem: every compact, connected Riemann surface of *genus* ≥ 2 is a quotient of the unit disk. (Genus 1 compact surfaces are quotients of $\mathbb{C}$, and genus 0 surfaces are the Riemann sphere $\mathbb{P}^1$.)

Riemann's existence theorem: every compact, connected Riemann surface admits a non-constant meromorphic function, so is a covering

Riemann-Hurwitz formula for genus of covering-space in terms of *ramification* over the base space.

Riemann-Roch theorem

Gamma function (Euler's integral) $\Gamma(s) = \int_0^\infty e^{-t}\, t^s\, \frac{dt}{t}$, Stirling's asymptotic formula

Riemann's zeta function $z(s) = \sum_{n=1}^\infty \frac{1}{n^s}$, Euler's product $z(s) = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}}$,

Jensen's formula counting zeros

Phragmén-Lindelöf theorem

Hadamard's three-circle theorem

Riemann's Explicit Formula

Topics from several complex variables...
