"Time change":
"DST" ends --
class will start 1 hr later
as of Monday!

30 Oct 2020
8/10x

⊐ notes 06 about (fixed-pt thm ⟹)

Implicit & Inverse fcns for smooth
⟹ real-diff.

↪ Impl. & Inv. fcn fn holo fcns

(continuing study of "which naturally-arising fcns are holo?!?" : "lots")

Saw (in context of $\Gamma(s)$: many $\int$'s of $F(t,z)$ w/ F holo in z on some nbd of $\mathbb{C}$ not nec. all of $\mathbb{C}$ $\Rightarrow$ dt are holo)

Sim: f holo, $f'(z_0) \neq 0$, then locally "near" $w_0 = f(z_0)$, $\exists$ f^{-1} to f & f^{-1} holo"

Ex: $f(z) = z^n$ $\quad f^{-1}$ ∃: $\sqrt[n]{w}$ $\quad n = 2,3,4,\ldots$

where $f'(z) \neq 0$: $f'(z) = nz^{n-1}$:

near

away from $z = 0$ ✓

+ Techn.: ?! $\exists$ $z_1, \dots, z_\ell$ distinct st

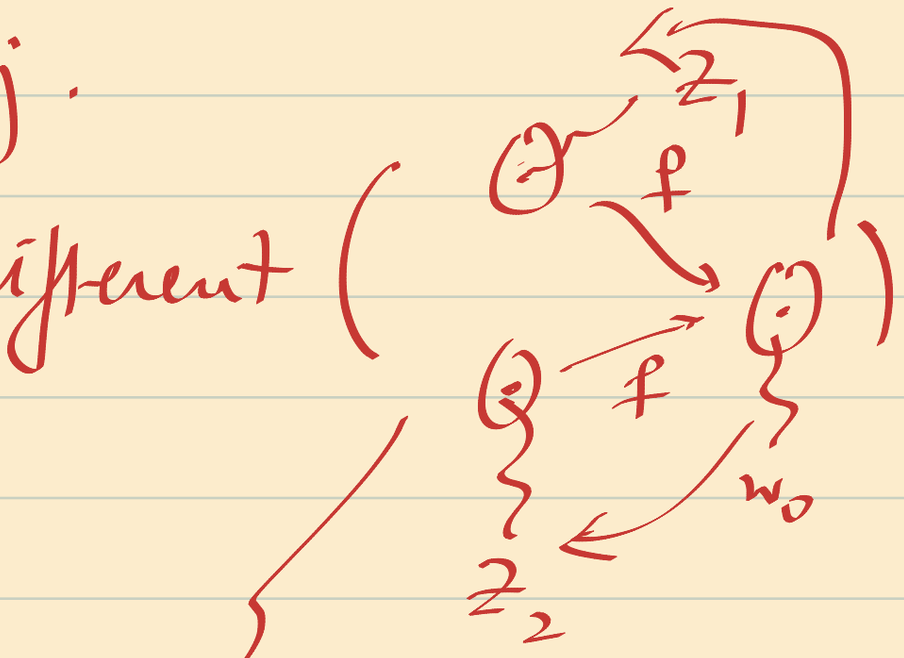
$$f(z_j) = w_0 \text{ (same)}, \forall j.$$

Thm, for each z_j, $\exists$ different

g_j hol'o st.

$$g_j(f(z_j)) = z_j$$

& $f(g_j(w_0)) = w_0$



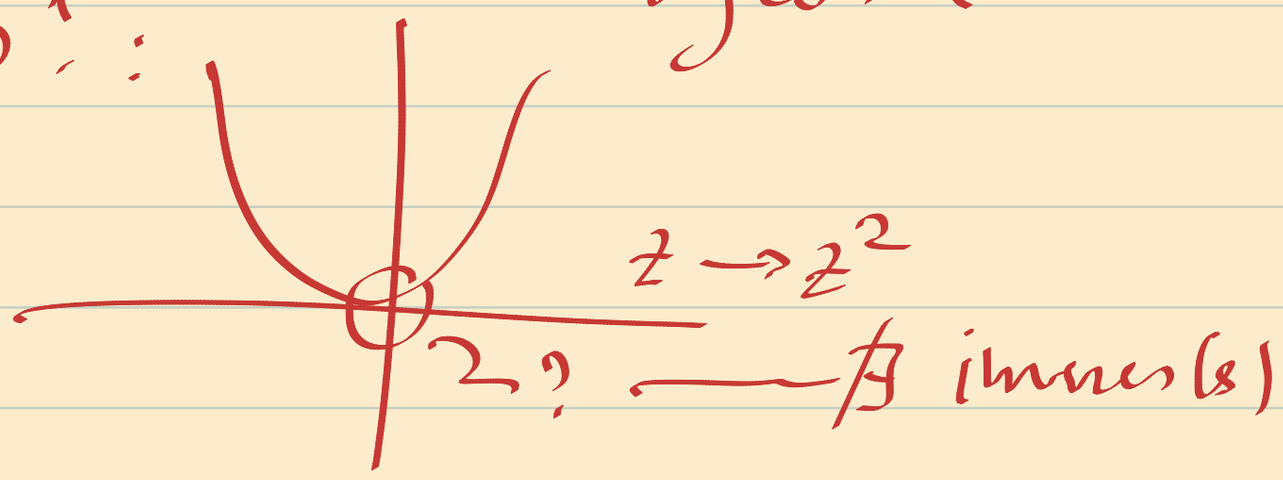
eg., $z \longrightarrow z^n$

(two-sided inverse)

(evidently!)

bijection

when $f'(z_j) \neq 0$!



Recall: g is left-inverse of f

when $g \circ f = id$

– right-inverse when $f \circ g = id$

(contrast to groups)

E.g., given n n^{th} roots $z_1, \dots, z_n$ of w_0 (n no's, 1 no.),

$\exists$ n hdlo (n^{th}-root) fcns $g_1, \dots, g_n$

def'd on same nbd of w_0 st.

$g_j(w_0) = z_j$ } "initial value"

Yes, $g_j = \underbrace{(z \to z^n)}_{f}$ || nbd of z_j (not all nbds of other z_k's)

E.g., $z \longrightarrow e^z$ has local inverse fcn(s)

many-to-one

* for $z_0 \longrightarrow w_0 = e^{z_0}$, also $z_0 + 2\pi i n \longrightarrow w_0$, $n \in \mathbb{Z}$

So, $\forall\ z_n = z_0 + 2\pi i n$, $\exists$ "$\log_n$" $(= \exp^{-1})$

st on small-enough nbd of w_0,

locally $\boxed{(\log_n)(e^z) = z}$ } fcn

& $\log_n(w_0) = z_n$ } numerical match at one pt.

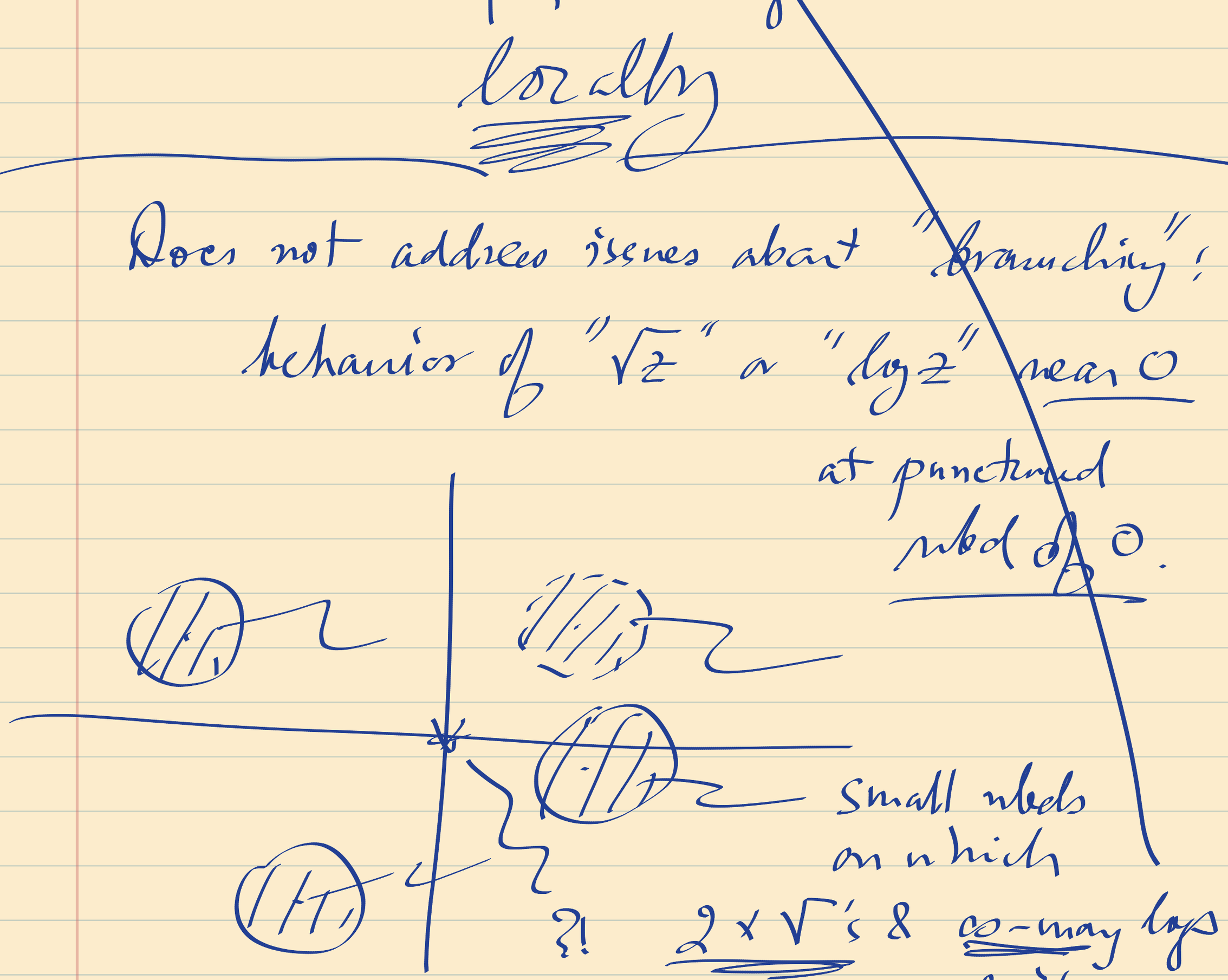
well-def'd, manifold

locally

Does not address issues about "branching"!

behavior of "$\sqrt{z}$" or "$\log z$" near 0

at punctured nbd of 0.



"Implicit"

Sim., even when ∄ fml "$w = f(z)$",

but only "joint":

$$z^5 + zw + w^5 = 1$$

(quintics not solvable by radicals) ⟹ cannot write as $w = f(z)$ nor $z = f(w)$ —.

$P(z,w) = 0$ ⟹ given z_0, w_0 st.

$P(z_0, w_0) = 0$

holo in each var

& real-diff, etc. (notes)

&

$P_w(z_0, w_0) \neq 0$

(divide?!)

⟹ ∃ holo $g(z)$ st.

$P(z, g(z)) = 0$ } fn

& $g(z_0) = w_0$ } base-pt

chain rule

no "fml"

extrapolates z, w in nbd(s) of z_0, w_0

numerical

f.p. thm for contractive maps

$\Longrightarrow$ inverse fct thm for C^∞

Inverse $\Longrightarrow$ implicit ✓ for C^∞ (clarify)

Lemma (see notes) if the real-deriv

of real-diff $f: \mathbb{C} \to \mathbb{C}$ is $\mathbb{C}$-linear

$\wr\wr$ (not just $\mathbb{C}$-linear)

$\mathbb{R}^2$

— then f is cx-diff. (& conversely)

Recall $f: \overset{\text{open } U}{\mathbb{R}^m} \to \mathbb{R}^n$ is $\mathbb{R}$-diff @ x_0

when $\exists$ linear $Df(x_0): \underset{\text{"}}{\mathbb{R}^m} \to \underset{\Rightarrow}{\mathbb{R}^n}$

(tangent spaces to U & $\mathbb{R}^n$)

s.t.

$$f(x_0+h) = f(x_0) + Df(x_0)(h) + o(h)$$

small in $\mathbb{R}^n$ — order 0 — 1st-order — Landau's little-oh

"as $|h| \to 0$" — "smaller"

(NB, no $\div h$ nor $\div |h|$...)

To say that $f(h)$ is $o(h)$ for $h \to 0$, $h \in \mathbb{R}^m$

means $\lim_{|h| \to 0} \frac{1}{|h|} \cdot g(h) = 0$ ($g(h)$)

$|h| \to 0$ &/or $h \to 0$ in $\mathbb{R}^n$

Does not say how fast.

Eg., Does not say $\exists C$ st $|g(h)| \leq C \cdot |h|^2$

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2 more (important) des. integral examples:

① (Fourier transform!) $\int_{-\infty}^{\infty} e^{-2\pi i t x} e^{-\pi x^2} dx$ } constants are "balanced"

compute

$\| \; \int_{-\infty}^{\infty} e^{itx} e^{-x^2} dx$ — Gaussian

$\| \; e^{-\pi t^2}$

$e^{-\pi x^2} = e^{-\pi x^2}$ ▽ //

not trivial!

Can ch. vbls

Heuristic near-proof:

$$\int_{-\infty}^{\infty} e^{-(2\pi i t x}\, e^{-\pi x^2}\, dx$$

$$= \int_{-\infty}^{\infty} e^{-\pi(x^2 + 2itx)}\, dx$$

$$= \int_{-\infty}^{\infty} e^{-\pi\left((x+it)^2 + t^2\right)}\, dx$$

complete the $\square$

$$= e^{-\pi t^2} \left[\int_{-\infty}^{\infty} e^{-\pi (x+it)^2}\, dx \right]$$

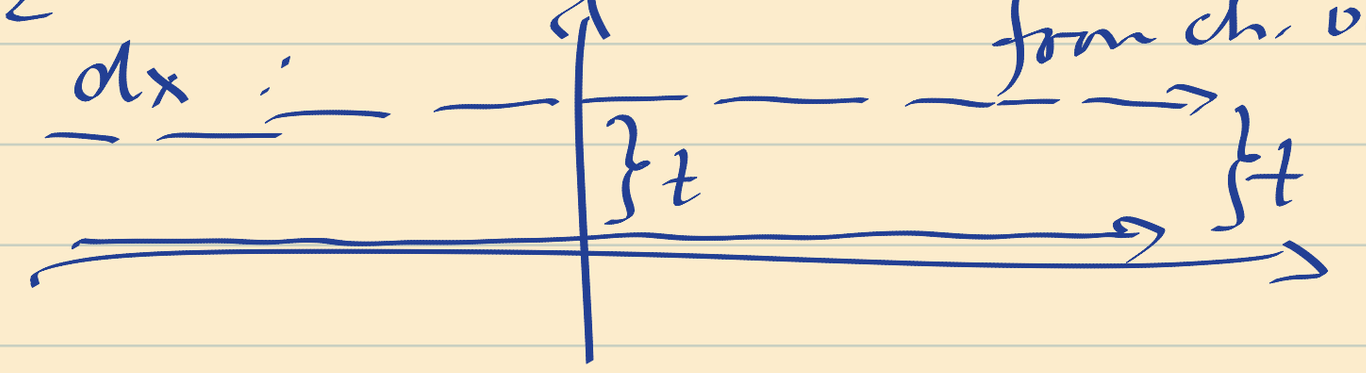
eval ?!

fine

??? Replace x by $x(+it)$: ch. vbles, & IR ??

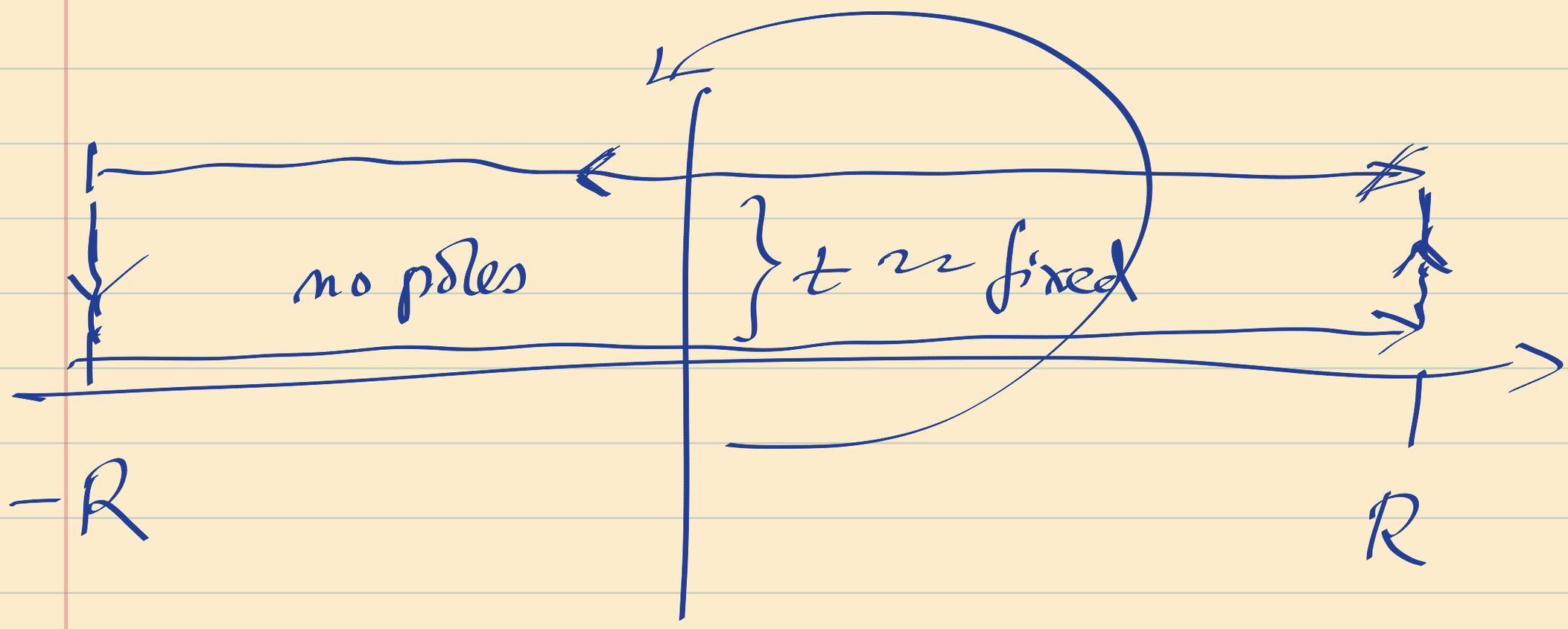
set $\int_{-\infty}^{\infty} e^{-\pi x^2}\, dx \overset{!!}{=} 1$ — !!

?. $\int_{-\infty+it}^{+\infty+it} e^{-\pi x^2}\, dx$ — from ch. vbles

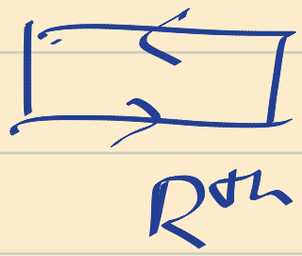


$(t > 0)$

Use Res. Thm to show two $\int$'s are same !



@ Res. Thm: $\int \approx 0$ ✓



! @ Aux arcs estimated (yon!!!) $\longrightarrow 0$ as $R \to \infty$ "

exp'ls can blow up vertically, but vertical limited by fixed t, ... ✓

$\Longrightarrow$ $\lim_R \left(\int_{bot} + \int_{top} \right) = \int_{-\infty+it}^{\infty+it}$ (horiz decay of Gaussian: ☺)

Res $\not\parallel$ R

0

bot: $-R$ to R

top: R to $-R$

$\int_{-\infty+it}^{\infty+it} = \int_{-\infty}^{\infty}$ ☺

$-\infty$ to ∞ ⟹ Two $\int$'s $\to$ equal

obj

$$\hookrightarrow > \gamma_{cs}: \int_{-\infty+it}^{+\infty+it} e^{-\pi x^2}\,dx \overset{\checkmark}{=} \int_{-\infty}^{\infty} e^{-\pi x^2}\,dx$$

Do

$$e^{-\pi x^2} = e^{-\pi x^2} \checkmark \quad //$$