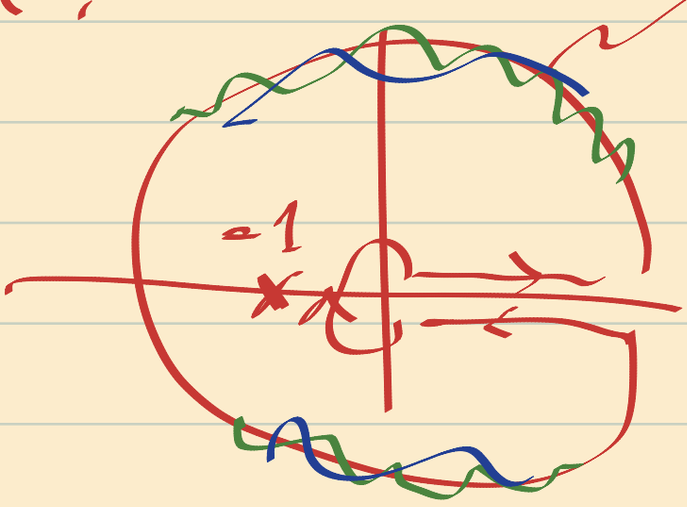
3 discussion 03 ✓ 13 Nov 2020

If you want critique ... 870x
get it to me soonish
(return all by at-worst Wed —)
Exam 03 a week from today ?

More about $\int_0^\infty \frac{x^s}{1+x}\,dx$ — family

✓ Abs. conv for $-1 < \mathrm{Re}(s) < 0$

Recall: (Residues)



to understand aux arcs' trivial est.
very nice to have

avoid? if possible

Then, comp'n = for real s, but/and
set expression $\left(\frac{\pi}{\sin \pi s}\right)$

$|i^i| = 1$?? NO

$|x^{\text{real}^2}| = 1$ — reflux ✓

$= \text{hdo}$ (way from poles)

$+ \int_0^\infty \frac{x^s}{1+x}\,dx = \text{hdo}$ ✓

agree on set $(-1,0)$ has acc. pts

so by Id. Princ. $\int_0^\infty \frac{x^s}{1+x}\,dx = \frac{\pi}{\sin \pi s}$ ✓

$\forall s \in \mathbb{C}$ st ↑ abs. conv ($\Rightarrow$ hdo)

Variants (∃ small trick):

$$\int_0^\infty \frac{x^s}{1+x^4}\,dx \overset{!?!}{=\!=\!=}$$

∃ 4 residues & messy — simplify? ✗

cvgce ✓

(maybe $s \in \mathbb{R}$, for easy aux arc estimates ✓)

↳ Replace x by $x^{1/4}$ "ch. vbles" to return to previous

$$= \int_0^\infty \frac{x^{5/4}}{1+x}\cdot\left(\frac{1}{4}\right)\frac{1}{x^{3/4}}\cdot dx$$

(∃! res. ✓ ☺)

⟸ easy (no $\mathfrak{D}'$s / no $e^{2\pi i \frac{1}{5}}$)

$$= \frac{1}{4}\int_0^\infty \frac{x^{\frac{s-3}{4}}}{1+x}\,dx$$

$$= \frac{1}{4}\,\frac{\pi}{\sin\pi\left(\frac{s-3}{4}\right)}$$ ✓ ☺

Sim $4 \rightsquigarrow 100$ ✓

$4 \rightsquigarrow$ real ✓ (real $\gg$ 1 for cvgce)

$4 \rightsquigarrow$ ex no. ?! : not literally of ↳ to ch. vbles ⟶ ☹

$$\int_0^\infty \frac{x^\alpha}{1+x^\beta}\,dx\,,\quad \alpha,\beta\in\mathbb{C}\,,\ \text{ch vbles ???}$$

(cvgce)

$x \mapsto x^{1/\beta}$???!?!

Anticipate applic of Id. Principle:

α, β real (& on axis) & at the end

Idiot-proof!" "Identity Princ."

"f holo on $\mathbb{C}$"

ex f entire & takes values in $\mathbb{H}$ (= "H") (upper $\mathbb{C}$-plane)

open

Claim f const. (Liouville??)

not absent

Use "(inverse) Cayley map" open mapping

(a very special/important "conformal mapping (later): angle-preserving)

3 other normalizations

$C(z) = \frac{z-i}{z+i}$ $\left(= \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}(z)\right)$

linear fraction/Möbius transformation (later)

$=1$

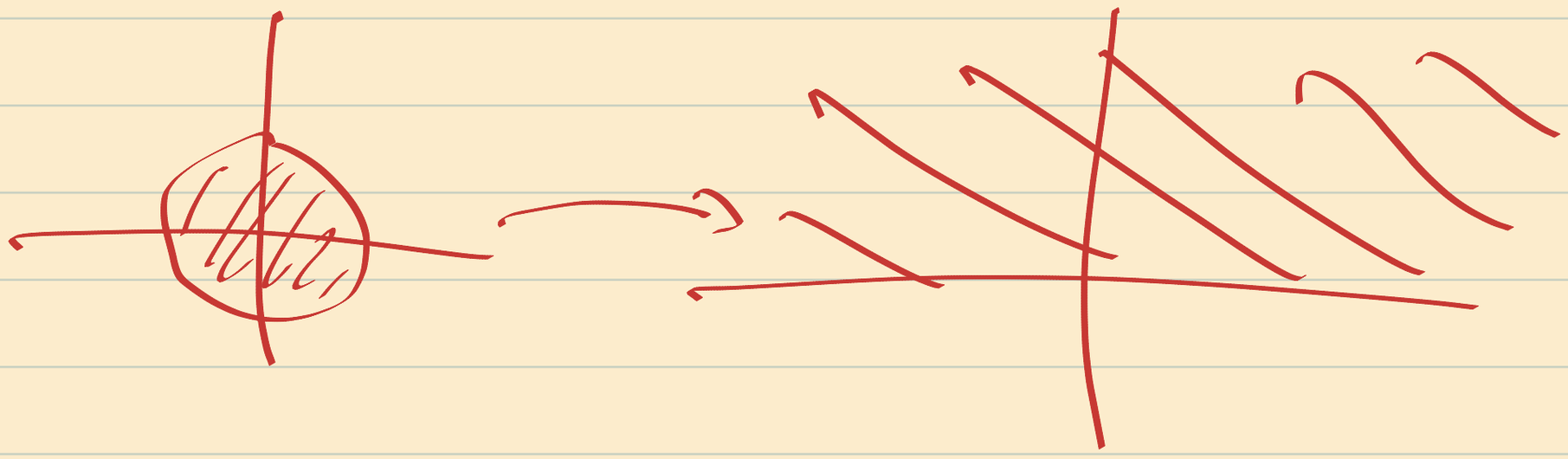
$C(z) = \frac{z\theta i}{\theta i z + 1}$

$C(z) = \frac{z+i}{iz+1}$

C: unit disk $\longrightarrow$ upper half-plane

analytic $\mathcal{O}(\mathbb{D}) \longrightarrow \mathcal{O}(\mathbb{H})$ (wiki "Sutterlin")

C :



$\tilde{C}^{-1}$:

(will recur endlessly!)

Here

$\tilde{C}^{-1} \circ f$ takes values in $\mathcal{D}$ = disk ;

$\tilde{C}^{-1} \circ f$ = bdd

$\Downarrow$ Liouville

const. ✓

$\tilde{C}^{-1} \circ f$ takes values in $\mathcal{H}$

$\Longrightarrow C \circ (\tilde{C}^{-1} \circ f) \overset{\text{assoc}}{=} (C \circ \tilde{C}^{-1}) \circ f$

$= \text{id} \circ f$

$= f$ ✓

const.

const.

Special : if f (entire) maps to $U \subsetneq \mathbb{C}$ & $\exists \subsetneq \mathbb{C}$ (what kind???)

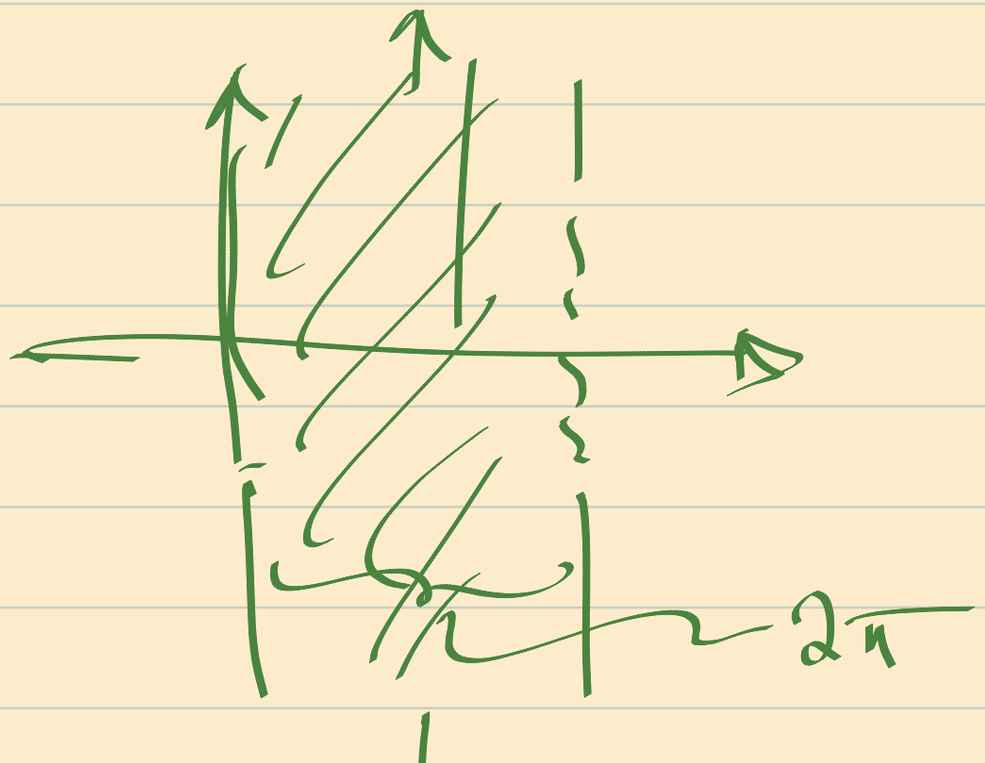
φ st $\varphi(U) = \text{bdd} \subseteq \mathbb{C}$, then $\varphi \circ f = \text{bdd}$

(invertible) bdd

so const. by Liouville ✓ ∎

⇝ "value distn theory:
how many values can
entire fcn omit?" (Picard thms ~1890??)
(later!!)

Ex: f entire, & takes values in "strip"/"band"
(wolog can translate & rotate)



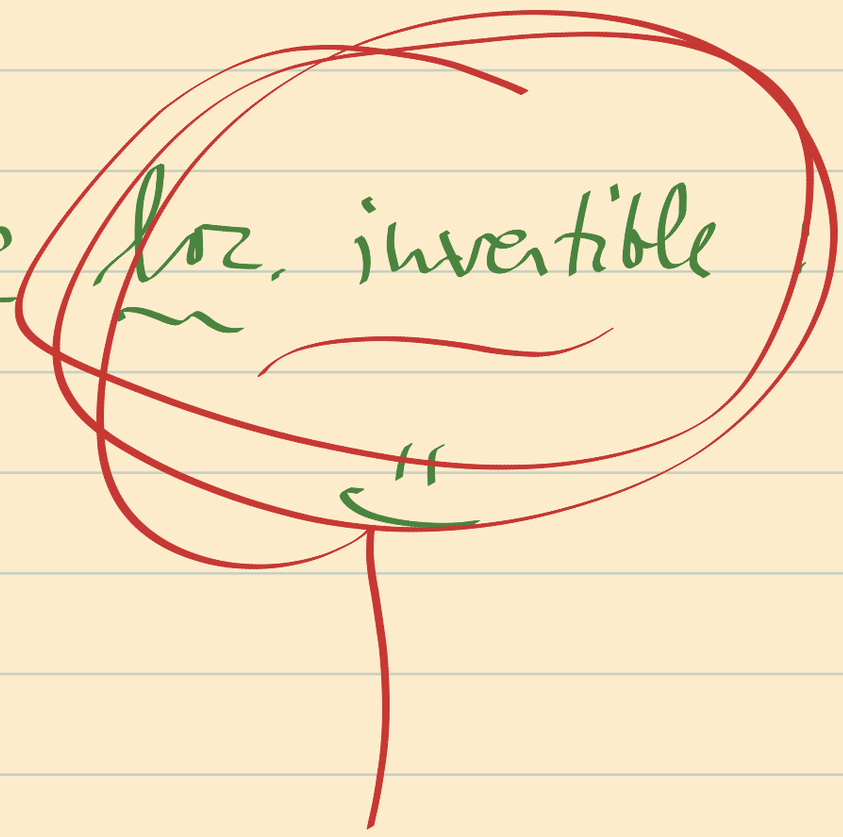
Claim: f constant (Liouville — (or maybe Picard))
? of is bdd?
bdd ⟵ "exp!"

$e^{f(z)}$ takes values (omit) in disk rad. 2π
≡ bdd ⟹

$\Rightarrow$ $e^{f(z)}$ is const. (Liouville) ;

* exp not invertible — (so if say it is, ... cred?!?)

But it is loc. invertible ∃ log.



≡ on suff small nbds of $z_0 \neq 0$ —

$\Rightarrow$ $f(z) \underset{\text{loc.}}{=} \log\left(\underbrace{e^{f(z)}}_{\text{const.}}\right)$

(loc) const.

f entire & "loc const."

e.g., on nbd U of z_0, $f(z) \equiv c$ on U

Then $f(z) = c$ on U (∃ accpts)

both hol on $\mathbb{C}$,

so by Id. Princ. ≡ ⚠ (use $\mathbb{C}$ = conn.)

Hmwk #1 : (~ version of Reflection :

f holo

≈

f holo on open disk, ext C^0 to closed disk,

& $f(\text{circle}) \subseteq \text{circle}$,

then

ext. f by $\overline{f(\overline{z})}$

$$f(1/z) = \frac{1}{\overline{f(\overline{z})}}$$

by - version gives this

f holo in $\mathbb{H}^{(\text{open})}$

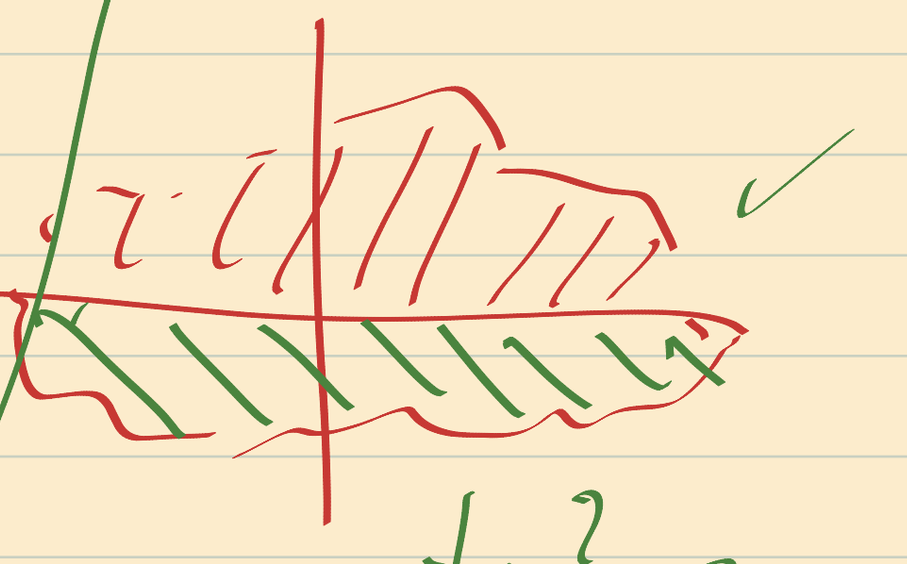
& ext to C^0 on closed $\frac{1}{2}$-plane

& $f(\mathbb{R}) \subseteq \mathbb{R}$

then $f(z) = \overline{f(\overline{z})}$

def's extn of f to $\mathbb{C}$ (in simple case)

More complicated possible



top?

$z \to \overline{f(\overline{z})}$ is holo (!)

See notes