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## Complex analysis examples discussion 01

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[01.1] Express the two values for  $\sqrt{i}$  in terms of radicals.

The definition of square root here is that  $(a+bi)^2 = i$ . Multiplying out gives  $(a^2 - b^2) + 2abi = i$ . Separating into real and imaginary parts gives  $a^2 - b^2 = 0$  and  $2ab = 1$ . From the first of these,  $b = \pm a$ , and substituting into the second gives  $\pm 2a^2 = 1$ . Thus,

$$\pm\sqrt{i} = \pm\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$$

Another approach, making more use of the geometry via the Euler identity  $e^{ix} = \cos x + i \sin x$ , is as follows, although a careful version of this is more verbose in this context. First, using that Euler identity,  $e^{ix} = i$  exactly for  $x = \frac{\pi}{2} + 2\pi n$  for integer  $n$ . Then  $(e^{iy})^2 = i$  implies  $e^{2iy} = e^{i(\frac{\pi}{2} + 2\pi n)}$  exactly for integers  $n$ . That is, again by Euler's identity,

$$2iy = \frac{\pi}{2} - 2\pi n \quad (\text{for integers } n)$$

and

$$\pm\sqrt{i} = e^{i(\frac{\pi}{4} + \pi n)} \quad (\text{for integers } n)$$

Since  $e^{\pi in} = 1$  exactly for integers  $n$ , the complete list of *distinct* values is

$$\pm\sqrt{i} = e^{i(\frac{\pi}{4})}, e^{i(\frac{\pi}{4} + \pi)} = \cos \frac{\pi}{4} + i \sin \frac{\pi}{4}, -\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right) = \pm\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$$

[01.2] Determine all values of  $i^i$ .

For this example, there is no alternative but to use Euler's identity, to raise complex numbers to the  $i^{\text{th}}$  power. That is, there is no alternative algebraic or limiting definition of  $\alpha^i$ , unlike the case of integer, rational, or real exponents. From Euler's identity,

$$i = e^{\frac{\pi i}{2} + 2\pi in} \quad (\text{for every integer } n)$$

Then

$$i^i = e^{(\frac{\pi i}{2} + 2\pi in) \cdot i} = e^{-\frac{\pi}{2} - 2\pi n} \quad (\text{for every integer } n)$$

It is striking that the outcome is a collection of real numbers approaching  $0^+$  at one end and going to  $+\infty$  at the other.

[01.3] Derive the usual formula for  $\sin(z+w)$  by using  $e^z$ .

Use  $\sin z = \frac{e^{iz} - e^{-iz}}{2i}$  and  $\cos z = \frac{e^{iz} + e^{-iz}}{2}$ , which hold for complex  $z$ , as well, and also the opposite formula  $e^{iz} = \cos z + i \sin z$ . Also note that  $\cos z$  is *even*, while  $\sin z$  is *odd*. Then the basic property  $e^{z+w} = e^z \cdot e^w$  and elementary algebra give the desired identity:

$$\begin{aligned} \sin(z+w) &= \frac{e^{i(z+w)} - e^{-i(z+w)}}{2i} = \frac{e^{iz} e^{iw} - e^{-iz} e^{-iw}}{2i} \\ &= \frac{(\cos z + i \sin z)(\cos w + i \sin w) - (\cos z - i \sin z)(\cos w - i \sin w)}{2i} \\ &= \frac{2i \sin z \cos w + 2i \cos z \sin w}{2i} = \sin z \cos w + \cos z \sin w \end{aligned}$$

[01.4] Express  $\cos 5x$  as a polynomial in  $\cos x$  and  $\sin x$ .

Again use  $\sin z = \frac{e^{iz} - e^{-iz}}{2i}$  and  $\cos z = \frac{e^{iz} + e^{-iz}}{2}$ ,  $e^{iz} = \cos z + i \sin z$ , and the even-ness of  $\cos x$  and odd-ness of  $\sin x$ :

$$\begin{aligned} \cos 5x &= \frac{e^{5ix} + e^{-5ix}}{2} = \frac{(e^{ix})^5 + (e^{-ix})^5}{2} = \frac{(\cos x + i \sin x)^5 + (\cos x - i \sin x)^5}{2} \\ &= \frac{\sum_{j=0}^5 \binom{5}{j} \cos^j x (i \sin x)^{5-j} + \sum_{j=0}^5 \binom{5}{j} \cos^j x (-i \sin x)^{5-j}}{2} \\ &= \sum_{\text{even } j=0}^5 \binom{5}{j} \cos^j x (i \sin x)^{5-j} = \cos^4 x + 10 \cos^2 x \sin^3 x + 4 \cos x \sin^4 x \end{aligned}$$

[01.5] By mere algebra, write a power series expansion near  $z = 0$  for

$$f(z) = \frac{1}{(z-1)(z-2)}$$

This has to allow use of the geometric series expansion  $\frac{1}{1-w} = 1 + w + w^2 + w^3 + \dots$  for  $|w| < 1$ . It is convenient to separate the two factors of the denominator by a partial fractions expansion, as

$$\frac{1}{(z-1)(z-2)} = \frac{-1}{z-1} + \frac{1}{z-2}$$

The coefficients can be found by looking at the asymptotics as  $z \rightarrow 1$  and  $z \rightarrow 2$ , for example. Thus,

$$\begin{aligned} \frac{1}{(z-1)(z-2)} &= \frac{-1}{z-1} + \frac{1}{z-2} = \frac{1}{1-z} + \frac{-\frac{1}{2}}{1-\frac{z}{2}} = (1+z+z^2+\dots) - \frac{1}{2}(1+\frac{z}{2}+(\frac{z}{2})^2+\dots) \\ &= \sum_{n=0}^{\infty} \left(1 - \frac{1}{2^n}\right) z^n \end{aligned}$$

[01.6] Determine the radius of convergence of  $\sum_{n \geq 1} \frac{3^n}{n(n+1)(n+2)} z^n$ .

The ratio test succeeds:

$$\frac{3^n/n(n+1)(n+2)}{3^{n+1}/(n+1)(n+2)(n+3)} = \frac{1/n}{3/(n+3)} = \frac{n+3}{3n} = \frac{1}{3} + \frac{1}{n} \rightarrow \frac{1}{3} \quad (\text{as } n \rightarrow +\infty)$$

so the radius of convergence is  $1/3$ .

[01.7] Determine the radius of convergence of  $\sum_{n \geq 1} \frac{n!}{n^n} z^n$ .

The ratio test succeeds:

$$\frac{n!/n^n}{(n+1)!/(n+1)^{n+1}} = \frac{1/n^n}{(n+1)/(n+1)^{n+1}} = \left(1 + \frac{1}{n}\right)^n \rightarrow e \quad (\text{as } n \rightarrow +\infty)$$

by one characterization of the number  $e$ . That is, the radius of convergence is  $e$ .

[01.8] For two complex numbers  $a, b$ , with  $b$  not a non-positive integer, show that the radius of convergence of

$$\sum_{n \geq 0} \frac{a(a+1)(a+2) \dots (a+n-1)(a+n)}{b(b+1)(b+2) \dots (b+n-1)(b+n)} z^n$$

is at least 1.

If  $a$  happens to be a negative integer, the sum terminates, and is a polynomial, so has radius of convergence  $+\infty$ .

For  $a$  not a negative integer, the ratio test succeeds:

$$\begin{aligned} \frac{a(a+1)(a+2) \dots (a+n-1)(a+n)}{b(b+1)(b+2) \dots (b+n-1)(b+n)} &/ \frac{a(a+1)(a+2) \dots (a+n)(a+n+1)}{b(b+1)(b+2) \dots (b+n)(b+n+1)} \\ &= \frac{b+n+1}{a+n+1} = \frac{1 + \frac{b+1}{n}}{1 + \frac{a+1}{n}} \longrightarrow 1 \quad (\text{as } n \rightarrow +\infty) \end{aligned}$$

so the radius of convergence is 1 in this case.

[01.9] From the very definition of convergence, show that when the partial sums of a series  $a_1 + a_2 + \dots$  are *bounded*, and when the elements of the sequence  $\{b_n\}$  are *positive* (real) and *go to 0 monotonically*, then the series  $\sum a_n b_n$  converges.

In fact, we show that the partial sums of  $\sum_n a_n b_n$  form a *Cauchy* sequence, so, by the *completeness* of  $\mathbb{C}$ , the infinite sum converges. That is, we prove that the *tails*  $\sum_{\ell=m}^n a_\ell b_\ell$  have the property that, given  $\varepsilon > 0$ , there is  $N$  such that  $|\sum_{\ell=m}^n a_\ell b_\ell| < \varepsilon$  for all  $m, n \geq N$ .

To this end, we rearrange the tails by *partial summation*: letting  $A_n = a_1 + \dots + a_n$  and  $B_n = b_1 + \dots + b_n$ ,

$$\begin{aligned} a_m b_m + a_{m+1} b_{m+1} + \dots + a_n b_n &= (A_{m+1} - A_m) b_m + \dots + (A_{n+1} - A_n) b_n \\ &= -A_m b_m + A_{m+1} (b_m - b_{m+1}) + A_{m+2} (b_{m+1} - b_{m+2}) + \dots + A_n (b_{n-1} - b_n) + A_{n+1} b_n \end{aligned}$$

Taking absolute values, letting  $C$  be a bound for the  $|A_n|$ , using  $b_j > b_{j+1} > 0$ ,

$$\begin{aligned} &\left| -A_m b_m + A_{m+1} (b_m - b_{m+1}) + \dots + A_n (b_{n-1} - b_n) + A_{n+1} b_n \right| \\ &\leq C b_m + C (b_m - b_{m+1}) + C (b_{m+1} - b_{m+2}) + \dots + C (b_{n-1} - b_n) + C b_n = C b_m \longrightarrow 0 \end{aligned}$$

since  $b_m \rightarrow 0$ . Thus, the partial sums are Cauchy, and the infinite sum converges.

[01.10] Show that the function  $f(z) = \sum z^n/n^2$  on the open disk  $|z| < 1$  extends to a *continuous* function on the *closed* unit disc.

Since the power series has radius of convergence 1, it gives an infinitely-differentiable, hence continuous, function on the *open* disk. There are several possible arguments to prove continuity on the *closed* disk, but in any case it is important to note that Abel's theorem about *non-tangential* approach does not instantly prove this, since every neighborhood of a point on the boundary circle contains points *outside* the non-tangential approach regions address.

A general argument that applies here is that a *uniform limit of continuous functions is continuous*. The finite subsums  $\sum_{n \leq N} z^n/n^2$  are *polynomials*, so certainly continuous. The differences are easily bounded, with  $M \leq N$ , with  $|z| \leq 1$ , by

$$\left| \sum_{1 \leq n \leq M} \frac{z^n}{n^2} - \sum_{1 \leq n \leq N} \frac{z^n}{n^2} \right| \leq \sum_{M < n \leq N} \frac{1}{n^2}$$

Since  $\sum_{n \geq 1} 1/n^2$  converges, given  $\varepsilon > 0$  there is  $M_o$  such that  $\sum_{M < n \leq N} \frac{1}{n^2} < \varepsilon$  for all  $M_o \leq M \leq N$ . Thus, the partial sums of  $\sum_n z^n/n^2$  for  $|z| \leq 1$  form a Cauchy sequence in sup norm, so converge to a continuous function on the closed disk.