(*September 24, 2014*)

Complex analysis examples discussion 02

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/˜garrett/

[This document is http://www.math.umn.edu/˜garrett/m/complex/examples_2014-15/cx_discussion_02.pdf]

[02.1] Parametrize counter-clockwise a circle γ of radius $r > 0$ centered at z_o, and *directly* compute $\int_\gamma (z - z_o)^n \, dz$ for all positive and negative integers n.

Such a path can be parametrized as $\gamma(t) = z_o + re^{it}$ for $0 \le t \le 2\pi$. Then

$$\int_\gamma (z - z_o)^n \, dz \;=\; \int_0^{2\pi} (re^{it})^n \, d(re^{it}) \;=\; \int_0^{2\pi} (re^{it})^n \, ire^{it} \, dt$$

$$= \; ir^{n+1} \int_0^{2\pi} e^{i(n+1)t} \, dt \;=\; \begin{cases} \Big[it\Big]_0^{2\pi} & = \; 2\pi i \quad (\text{for } n = -1) \\ \Big[\dfrac{ir^{n+1} \cdot e^{i(n+1)t}}{i(n+1)}\Big]_0^{2\pi} & = \; 0 \quad (\text{for } n \ne -1) \end{cases}$$

[02.2] Using only geometric series expansions, determine the Laurent expansion of $f(z) = 1/(z-1)(z-2)$ in the annulus $1 < |z| < 2$, and also in the annulus $|z| > 2$.

By partial fractions, for $1 < |z| < 2$, expanding geometric series,

$$\frac{1}{(z-1)(z-2)} \;=\; \frac{1}{z-2} - \frac{1}{z-1} \;=\; \frac{-\frac{1}{2}}{1 - \frac{z}{2}} - \frac{1}{z} \cdot \frac{1}{1 - \frac{1}{z}} \;=\; -\tfrac{1}{2}\Big(1 + \frac{z}{2} + (\frac{z}{2})^2 + \ldots\Big) - \frac{1}{z}\Big(1 + \frac{1}{z} + \Big(\frac{1}{z}\Big)^2 + \ldots\Big)$$

$$= \; -\tfrac{1}{2} - \sum_{n=1}^\infty (1 + \frac{1}{2^{n+1}}) \, z^{-n} \qquad (\text{in the annulus } 1 < |z| < 2)$$

For $|z| > 2$, the $1/(z-2)$ requires slightly different treatment:

$$\frac{1}{(z-1)(z-2)} \;=\; \frac{1}{z-2} - \frac{1}{z-1} \;=\; \frac{1}{z} \cdot \frac{1}{1 - \frac{2}{z}} + \frac{1}{z} \cdot \frac{1}{1 - \frac{1}{z}} \;=\; \frac{1}{z}\Big(1 + \frac{2}{z} + (\frac{2}{z})^2 + \ldots\Big) - \frac{1}{z}\Big(1 + \frac{1}{z} + \Big(\frac{1}{z}\Big)^2 + \ldots\Big)$$

$$= \; \sum_{n=1}^\infty (2^{n-1} - 1) \, z^{-n} \;=\; \sum_{n=2}^\infty (2^{n-1} - 1) \, z^{-n} \qquad (\text{in the annulus } 1 < |z| < 2)$$

[02.3] Determine the Laurent expansion of $f(z) = 1/(z-1)^4$ in the annulus $|z| > 1$, and also in the annulus $|z - 1| > 0$.

In $|z| > 1$,

$$\frac{1}{z-1} \;=\; \frac{1}{z} \cdot \frac{1}{1 - \frac{1}{z}} \;=\; \frac{1}{z} \cdot \Big(1 + \frac{1}{z} + \Big(\frac{1}{z}\Big)^2 + \ldots\Big) \;=\; \frac{1}{z} + \Big(\frac{1}{z}\Big)^2 + \ldots$$

Differentiating termwise three times gives

$$\frac{(-1)(-2)(-3)}{(z-1)^4} \;=\; \frac{(-1)(-2)(-3)}{z^4} + \frac{(-2)(-3)(-4)}{z^5} + \ldots + \frac{(-n)(-n-1)(-n-2)}{z^{n+3}} + \ldots$$

which simplifies to

$$\frac{1}{(z-1)^4} \;=\; \frac{1}{z^4} + \frac{2 \cdot 3 \cdot 4/6}{z^5} + \ldots + \frac{n(n+1)(n+2)/6}{z^{n+3}} + \ldots$$

In the annulus $|z-1|>0$, the given expression $f(z)=(z-1)^{-4}$ is already the Laurent expansion.

[02.4] Show that an entire function f satisfying $|f(z)| \le C\cdot(1+|z|)^{1/2}$ for some constant C is *constant*.

The argument is nearly identical to that of Liouville's theorem that *bounded* entire functions are constant. Namely, Cauchy's formula for the n^{th} power series coefficient c_n of f at 0, via a circle γ_R of radius R for any $R>0$, is

$$c_n \;=\; \frac{1}{2\pi i}\int_{\gamma_R}\frac{f(w)\,dw}{w^{n+1}} \;\le\; \text{length}(\gamma_R)\cdot\sup_{\gamma_R}\Big|\frac{f(w)}{w^{n+1}}\Big| \;\le\; \pi R\cdot\sup_{\gamma_R}\frac{C\cdot(1+R)^{\frac{1}{2}}}{R^{n+1}}$$

by the trivial estimate on the absolute value of a path integral. For $0<n\in\mathbb{Z}$ this goes to 0 as $R\to+\infty$, so all but the 0^{th} power series coefficient are 0. Since f is entire, it is represented on the whole plane by its power series, so is constant.

[02.5] Compute $\displaystyle\int_{-\infty}^{\infty}\frac{dx}{x^4+1}$.

First, the infinite integral is a limit of finite limits

$$\int_{-\infty}^{\infty}\frac{dx}{x^4+1} \;=\; \lim_{R\to+\infty}\int_{-R}^{R}\frac{dx}{x^4+1}$$

Note that the denominator has zeros at eighth roots of unity, namely, $\zeta=\zeta_8=e^{\pi i/4}$, $\zeta^3=e^{3\pi i/4}$, $\zeta^5=e^{5\pi i/4}$, $\zeta^7=e^{7\pi i/4}$. Let γ_R be the path from $-R$ to R along the real line, and then along the arc of the circle of radius R in the upper half-plane, from $+R$ back to $-R$. The integral over the arc is estimated via the trivial estimate:

$$\Big|\int_{\text{arc } R}\frac{dx}{x^4+1}\Big| \le \text{length}(\text{arc } R)\cdot\sup_{\text{on arc } R}\Big|\frac{1}{z^4+1}\Big| \le \pi R\cdot\frac{1}{(R-1)^4}$$

This goes to 0 as $R\to+\infty$. Thus, using the Residue Theorem, the original integral is

$$\int_{-\infty}^{\infty}\frac{dx}{x^4+1} \;=\; \lim_R\int_{\gamma_R}\frac{dz}{1+z^4} \;=\; \lim_R 2\pi i\;\text{Res}_{z=\zeta,\,\zeta^3}\,\frac{1}{1+z^4}$$

$$=\; 2\pi i\Big(\frac{1}{(\zeta-\zeta^3)(\zeta-\zeta^5)(\zeta-\zeta^7)}+\frac{1}{(\zeta^3-\zeta)(\zeta^3-\zeta^5)(\zeta^3-\zeta^7)}\Big)$$

recalling the convenient fact that the residue at z_o of $g(z)/(z-z_o)$ for g holomorphic at z_o is $g(z_o)$. This is

$$2\pi i\Big(\frac{1}{(\sqrt{2})(2\zeta)(i\sqrt{2})}+\frac{1}{(-\sqrt{2})(i\sqrt{2})(2i\zeta)}\Big) \;=\; \frac{\pi i}{2}\Big(\frac{1}{i\zeta}+\frac{1}{\zeta}\Big) \;=\; \frac{\pi\zeta^2}{2}\cdot\frac{1-i}{\zeta} \;=\; \frac{\pi}{2}\cdot\frac{1+i}{\sqrt{2}}\cdot(1-i) \;=\; \frac{\pi}{\sqrt{2}}$$

[02.6] Compute $\displaystyle\int_{-\infty}^{\infty}\frac{e^{itx}\,dx}{x^4+1}$ with real t.

As in the previous example, the infinite integral is a limit of finite limits

$$\int_{-\infty}^{\infty}\frac{e^{itx}\,dx}{x^4+1} \;=\; \lim_{R\to+\infty}\int_{-R}^{R}\frac{e^{itx}\,dx}{x^4+1}$$

The denominator has zeros at eighth roots of unity $\zeta=e^{\pi i/4}$, ζ^3, ζ^5, ζ^7. Let γ_R be the path from $-R$ to R along the real line, and then along the arc of the circle of radius R in the upper half-plane, from $+R$ back to $-R$. The integral over the arc is estimated via the trivial estimate:

$$\Big|\int_{\text{arc } R}\frac{e^{itx}\,dx}{x^4+1}\Big| \le \text{length}(\text{arc } R)\cdot\sup_{\text{on arc } R}\Big|\frac{e^{itz}}{z^4+1}\Big| \le \pi R\cdot\frac{e^{t\cdot-\text{Im}\,(z)}}{(R-1)^4}$$

For $t \geq 0$, this goes to 0 as $R \to +\infty$. Using the Residue Theorem, the original integral is

$$\int_{-\infty}^{\infty} \frac{e^{itx}\,dx}{x^4+1} \;=\; \lim_R \int_{\gamma_R} \frac{e^{itz}\,dz}{1+z^4} \;=\; \lim_R 2\pi i \;\mathrm{Res}_{z=\zeta,\,\zeta^3}\,\frac{e^{itz}}{1+z^4}$$

$$=\; 2\pi i\Big(\frac{e^{it\zeta}}{(\zeta-\zeta^3)(\zeta-\zeta^5)(\zeta-\zeta^7)} + \frac{e^{it\zeta^3}}{(\zeta^3-\zeta)(\zeta^3-\zeta^5)(\zeta^3-\zeta^7)}\Big) \qquad \text{(for } t \geq 0)$$

recalling the convenient fact that the residue at z_o of $g(z)/(z-z_o)$ for g holomorphic at z_o is $g(z_o)$. The denominators simplify somewhat:

$$(\zeta-\zeta^3)(\zeta-\zeta^5)(\zeta-\zeta^7) \;=\; (\sqrt{2})(2\zeta)(i\sqrt{2}) \;=\; 4i\zeta$$

and

$$(\zeta^3-\zeta)(\zeta^3-\zeta^5)(\zeta^3-\zeta^7) \;=\; 4\zeta$$

so the $t \geq 0$ case gives

$$2\pi i\Big(\frac{e^{it\zeta}}{4i\zeta} + \frac{e^{it\zeta^3}}{4\zeta}\Big) \;=\; \frac{\pi}{2\zeta}e^{it\zeta} + \frac{\pi i}{2\zeta}e^{it\zeta^3} \qquad \text{(for } t \geq 0)$$

For $t < 0$, replacing x by $-x$ in the original integral reduces to the previous case. That is,

$$\int_{-\infty}^{\infty} \frac{e^{itx}\,dx}{x^4+1} \;=\; \frac{\pi}{2\zeta}\Big(e^{i|t|\zeta} + ie^{i|t|\zeta^3}\Big)$$

[02.7] Compute $\displaystyle\int_0^\infty \frac{x\,dx}{1+x^3}$

As usual, the integral is the limit of finite integrals $\int_0^R$ as $R \to +\infty$. Let γ_R be the path from 0 to R along the real line, then counter-clockwise along the circle of radius R to $R \cdot e^{2\pi i/3}$, then back along the straight line to 0. This path is chosen because the integral from $R \cdot e^{2\pi i/3}$ to 0 is very simply related to the original:

$$\int_R^0 \frac{(e^{2\pi i/3}t)\;d(e^{2\pi i/3}t)}{1+(e^{2\pi i/3}t)^3} \;=\; -e^{4\pi i/3}\int_0^R \frac{t\;dt}{1+t^3}$$

The integral along the arc is easily estimate by the trivial estimate:

$$\Big|\int_{\mathrm{arc}\ R} \frac{z\;dz}{1+z^3}\Big| \;\leq\; \mathrm{length}(\mathrm{arc}\ R)\cdot \sup_{\mathrm{on\ arc}\ R}\Big|\frac{z}{1+z^3}\Big| \;\leq\; \frac{2\pi R}{3}\cdot\frac{R}{(R-1)^3}$$

which goes to 0 as $R \to +\infty$. The integral over γ_R can be evaluated by residues: for $R > 1$, there is a single singularity inside γ_R, at the sixth root of unity $\zeta = \zeta_6 = e^{\pi i/3}$. Noting that

$$z^3+1 \;=\; (z+1)(z^2-z+1) \;=\; (z+1)(z-\zeta)(z-\zeta^{-1})$$

and that $-e^{4\pi i/3} = \zeta$, we have

$$(1+\zeta)\int_0^\infty \frac{x\;dx}{1+x^3} \;=\; \lim_R \int_{\gamma_R} \frac{z\;dz}{1+z^3} \;=\; 2\pi i\;\mathrm{Res}_{z=\zeta}\frac{z}{1+z^3} \;=\; 2\pi i\frac{\zeta}{(\zeta+1)(\zeta-\zeta^{-1})}$$

so

$$\int_0^\infty \frac{x\,dx}{1+x^3} \;=\; 2\pi i \frac{\zeta}{(\zeta+1)^2\,(\zeta-\zeta^{-1})} \;=\; 2\pi i \frac{1}{(\zeta+1)(\zeta^{-1}+1)\,(i\sqrt{3})} \;=\; \frac{2\pi}{(\frac{1+i\sqrt{3}}{2}+1)(\frac{1-i\sqrt{3}}{2}+1)\sqrt{3}}$$

$$= \frac{2\pi}{(\frac{9}{4}+\frac{3}{4})\sqrt{3}} \;=\; \frac{2\pi}{3\sqrt{3}}$$

[02.8] Compute $\frac{1}{1}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\ldots$

Arrange to evaluate the infinite sum by residues, by using the function $2\pi i/(e^{2\pi iz}-1)$, which we will check has simple poles with residues 1 at integers, and for z bounded away from integers is *bounded*. Granting that for a moment, letting γ_T be a counter-clockwise path around the square with vertices $\pm T \pm iT$ with $T \in \frac{1}{2}+\mathbb{Z}$, by residues

$$\int_{\gamma_T} \frac{2\pi i}{e^{2\pi iz}-1}\cdot\frac{1}{z^4}\,dz \;=\; \sum_{0\le|n|<T} \mathrm{Res}_{z=n}\frac{2\pi i}{e^{2\pi iz}-1}\cdot\frac{1}{z^4} \;=\; \sum_{0<|n|<T}\frac{1}{n^4}+\mathrm{Res}_{z=0}\frac{2\pi i}{e^{2\pi iz}-1}\cdot\frac{1}{z^4}$$

Because of the division by z^4, the latter residue is visibly the coefficient of z^3 in the Laurent expansion of $2\pi i/(e^{2\pi iz}-1)$, which is determined by expanding $1/(e^z-1)$

$$\frac{1}{e^z-1} \;=\; \frac{1}{(1+z+\frac{z^2}{2}+\frac{z^3}{6}+\frac{z^4}{24}+\ldots)-1} \;=\; \frac{1}{z+\frac{z^2}{2}+\frac{z^3}{6}+\frac{z^4}{24}+\ldots}$$

$$= \frac{1}{z}\cdot\frac{1}{1+\Big(\frac{z}{2}+\frac{z^2}{6}+\frac{z^3}{24}+\ldots\Big)} \;=\; \frac{1}{z}\Big(1-\Big(\frac{z}{2}+\frac{z^2}{6}+\ldots\Big)+\Big(\frac{z}{2}+\frac{z^2}{6}+\ldots\Big)^2-\Big(\frac{z}{2}+\frac{z^2}{6}+\ldots\Big)^3+\ldots\Big)$$

The z^3 coefficient of the latter is

$$-\frac{1}{5!}+\Big(2\cdot\frac{1}{2!}\cdot\frac{1}{4!}+1\cdot\Big(\frac{1}{3!}\Big)^2\Big)-3\cdot\Big(\frac{1}{2!}\Big)^2\cdot\frac{1}{3!}+\Big(\frac{1}{2!}\Big)^4 \;=\; -\frac{1}{120}+\frac{1}{24}+\frac{1}{36}-\frac{1}{8}+\frac{1}{16}$$

Replacing z by $2\pi iz$ in that Laurent expansion, and multiplying from the $2\pi i$ from the numerator multiplies this by $(2\pi i)^4 = 16\pi^4$, giving

$$-\frac{16}{120}+\frac{16}{24}+\frac{16}{36}-\frac{16}{8}+\frac{16}{16} \;=\; -\frac{2}{15}+\frac{2}{3}+\frac{4}{9}-1 \;=\; \frac{-6+30+20-45}{45} \;=\; -\frac{1}{45}$$

Thus, still granting that everything works out, we have

$$\int_{\gamma_T}\frac{2\pi i}{e^{2\pi iz}}\,\frac{1}{z^4}\,dz \;=\; \sum_{0<|n|<T}\frac{1}{n^4}+\mathrm{Res}_{z=0}\frac{2\pi i}{e^{2\pi iz}-1}\cdot\frac{1}{z^4} \;=\; \sum_{0<|n|<T}\frac{1}{n^4}-\frac{1}{45}$$

Taking the limit, the integral goes to 0, so

$$0 \;=\; \lim_T \sum_{0<|n|<T}\frac{1}{n^4}-\frac{1}{45} \;=\; 2\cdot\sum_{n\ge1}\frac{1}{n^4}\;-\;\frac{1}{45}$$

giving the claimed result. For the other details:

The function $2\pi i/(e^{2\pi iz}-1)$ has no singularities unless the denominator is 0, which occurs exactly at integers. It is $\mathbb{Z}$-periodic, so to check that its residue at 0 is 1: as above,

$$\frac{2\pi i}{e^{2\pi iz}-1} = \frac{2\pi i}{(1+(2\pi iz)+\frac{(2\pi iz)^2}{2}+\ldots)-1} = \frac{2\pi i}{2\pi iz+\frac{(2\pi iz)^2}{2}+\ldots} = \frac{1}{z+\frac{2\pi iz^2}{2}+\ldots}$$
$$= \frac{1}{z}\cdot\frac{1}{1+\Big(2\pi i\frac{z}{2}+\ldots\Big)} = \frac{1}{z}\Big(1-\Big(\frac{2\pi iz}{2}+\ldots\Big)+\ldots\Big)$$

To check that this function is *bounded* for z away from integers, first observe that $|e^{2\pi iz}| \le \frac{1}{e}$ for $\mathrm{Im}(z) \ge \frac{1}{2\pi}$, and $|e^{2\pi iz}| \ge e$ for $\mathrm{Im}(z) \le -\frac{1}{2\pi}$. In both cases, $e^{2\pi iz}-1$ is bounded away from zero, so $2\pi i/(e^{2\pi iz}-1)$ is bounded.

For $|\mathrm{Im}(z)| \le \frac{1}{2\pi}$, again use periodicity, to reduce to the set where $|\mathrm{Im}(z)| \le \frac{1}{2\pi}$, $0 \le \mathrm{Re}(z) \le 1$, and $|z-0| \ge \frac{1}{2}$ and $|z-1| \ge 1$. This set is *compact*, and $|2\pi i/(e^{2\pi iz}-1)|$ is continuous on it, so is bounded. This completes the checking of the background details to make things work.
