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Complex analysis examples discussion 03

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/˜garrett/

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[03.1] For a bounded sequence of complex numbers c_n, prove that $\displaystyle\sum_{n=0}^\infty c_n \frac{z^n}{z^n+1}$ converges to a holomorphic function on $|z|<1$.

Each summand is holomorphic on $|z|<1$, because of the *quotient rule*, and that the numerator and denominator are polynomials, hence holomorphic.

To prove that the sum $\sum_n f_n$ of a sequence of holomorphic functions on $|z|<1$ is itself holomorphic, it suffices to prove that the convergence is *uniform on compacts*. The compact subsets of the open disk are all contained in compact disks $|z|\le r$ for $r<1$, so it suffices to consider just those sets $|z|\le r$.

Given $r<1$, there is large-enough N such that $r^n \le \frac{1}{2}$ for all $n\ge N$, for example taking $N\ge \frac{\log\frac{1}{2}}{\log r}$. For $|z|\le r$ and $n\ge N$,

$$\Big|\frac{z^n}{1+z^n}\Big| \;\le\; \frac{|z|^n}{1-\frac{1}{2}} \;\le\; 2r^n$$

Thus, given $0<r<1$, let N so that $r^n\le\frac{1}{2}$ for all $n\ge N$. Given $\varepsilon>0$, for $m,n\ge N$, with $|c_n|\le B$ for all n,

$$\Big|\sum_{m\le j<n} c_n\,\frac{z^j}{1+z^j}\Big| \;\le\; B\cdot\sum_{m\le j<n} 2r^j \;<\; B\cdot\sum_{m\le j<\infty} 2r^j \;\le\; B\frac{2r^m}{1-r}$$

Increasing N if necessary, this is smaller than ε. ///

There are several viable variant approaches. Among others: expanding the power series for each $z^n/(z^n+1)$, although one should be careful *not* to suggest that a sum of holomorphic functions on a disk is holomorphic on that disk, since $\sum_n c_n\,z^n$ can have arbitrary radius of convergence, while the summands $c_n\,z^n$ have infinite radius of convergence. Invocation of Morera's theorem also works here.

[03.2] Prove that $\displaystyle f(z)=\int_0^1 \frac{e^{tz}\,dt}{t^2+1}$ is holomorphic.

The simplest argument might be to invoke Morera's theorem after changing order of integration. The change of order is easily justifiable, since one is looking at a continuous function of two variables. That is, for each $t\in[0,1]$, the function $z\to\frac{e^{tz}}{t^2+1}$ is holomorphic, and the function of two variables is continuous. Thus, letting γ be a small triangle,

$$\int_\gamma\int_0^1 \frac{e^{tz}\,dt}{t^2+1}\,dz \;=\; \int_0^1\int_\gamma \frac{e^{tz}}{t^2+1}\,dz\,dt \;=\; \int_0^1 0\,dt \;=\; 0$$

by applying Cauchy's theorem to $z\to\frac{e^{tz}}{t^2+1}$. By Morera, $f(z)$ is continuous. ///

Another approach is to view the integral as a uniform limit of a sequence of finite (Riemann) sums, each of which is holomorphic, being a finite sum of holomorphic functions, and then invoke the holomorphy of uniform (on compacts) limits of holomorphic functions.

[03.3] Prove that $f(z) = \int_0^\infty \frac{e^{-tz}\,dt}{t^2+1}$ is holomorphic for $\mathrm{Re}(z) > 0$.

Using the previous example, it would suffice to show that the sequence of finite integrals

$$f_n(z) \;=\; \int_0^n \frac{e^{-tz}\,dt}{t^2+1}$$

converges uniformly to $f(z)$ for z in compact subsets of $\mathrm{Re}(z) > 0$, since these finite integrals are holomorphic functions, via Morera.

For fixed $\delta > 0$ and $\mathrm{Re}(z) \geq \delta$, for $N \leq m \leq n$,

$$\Big|f_m(z) - f_n(z)\Big| \;\leq\; \int_m^n \frac{e^{-t\delta}\,dt}{t^2+1} \;\leq\; \int_N^\infty e^{-t\delta}\,dt \;=\; \frac{e^{-N\delta}}{\delta}$$

This can be made smaller than a given $\delta > 0$ by taking N sufficiently large. ///

[03.4] Let f be a continuous, bounded real-valued function on $\mathbb{R}$, extending to a bounded, holomorphic function on the upper half-plane $\mathfrak{H}$. Show f is constant.

This is an invocation of the reflection principle, and then Liouville's theorem, as follows. For any real $a < b$, the hypothesis gives a continuous extension of f to $\mathfrak{H} \cup (a,b)$, and then by reflection to $\mathfrak{H} \cup (a,b) \cup \mathfrak{H}^{\text{cx conj}}$. This argument succeeds for every $a < b$, so f extends to the ascending union of these sets, namely, the whole complex plane.

The expression $\widetilde{f}(z) = \overline{f(\overline{z})}$ for the extension to the lower half-plane shows that (absolute value of) the extension has the same bound as the original function. Thus, the extension to $\mathbb{C}$ is bounded, and by Liouville is constant. ///

[03.5] Evaluate the Fourier transform $\int_{-\infty}^\infty e^{-itx} \cdot \frac{1}{(x+i)^s}\,dx$ for complex s with $\mathrm{Re}(s) > 1$, using the Gamma function.

My preferred approach to this, while not the shortest, nicely illustrates some important methodological and technical points.

Recall that the identity principle gives

$$\int_0^\infty e^{-uz}\,u^s\,\frac{du}{u} \;=\; z^{-s}\,\Gamma(s) \qquad\qquad (\text{for } \mathrm{Re}(z) > 0 \text{ and } \mathrm{Re}(s) > 0)$$

Using this identity in the problem at hand,

$$\int_{-\infty}^\infty e^{-itx}\,\frac{1}{(x+i)^s}\,dx \;=\; i^{-s}\int_{-\infty}^\infty e^{-itx}\,\frac{1}{(1-ix)^s}\,dx \;=\; i^{-s}\frac{1}{\Gamma(s)}\int_{-\infty}^\infty\int_0^\infty e^{-itx}\,e^{-u(1-ix)}u^s\,\frac{du}{u}\,dx$$

Changing the order of integration, *if justifiable*, would give

$$i^{-s}\frac{1}{\Gamma(s)}\int_0^\infty e^{-u}\,\Big(\int_{-\infty}^\infty e^{i(u-t)x}\,dx\Big)u^s\,\frac{du}{u}$$

The difficulty is that the inner integral is not at all convergent in a classical, pointwise sense. Thus, with hindsight, the interchange of integrals is not justifiable in classical terms.

Nevertheless, that integral should remind us of *Fourier Inversion*: for nice-enough functions,

$$f(x) \;=\; \frac{1}{2\pi}\int_{-\infty}^\infty e^{i\xi x}\Big(\int_{-\infty}^\infty e^{-i\xi u}\,f(u)\,du\Big)\,d\xi$$

In particular, there is an illuminating heuristic, or near-proof, for Fourier Inversion, involving the same not-classically-justifiable interchange of integrals:

$$\int_{-\infty}^{\infty} e^{i\xi x}\Big(\int_{-\infty}^{\infty} e^{-i\xi u}\, f(u)\, du\Big)\, d\xi \;=\; \int_{-\infty}^{\infty}\Big(\int_{-\infty}^{\infty} e^{i\xi(x-u)}\, d\xi\Big) f(u)\, du \qquad (?????)$$

Since we *know* that this should be $2\pi\cdot f(x)$, it must be that, in effect,

$$\int_{-\infty}^{\infty} e^{i\xi(x-u)}\, d\xi \;=\; 2\pi\cdot\delta(x-u) \qquad \text{(Dirac delta)}$$

Granting this heuristic for a moment, the integral at hand would become

$$2\pi\cdot i^{-s}\frac{1}{\Gamma(s)}\int_0^{\infty} e^{-u}\,\delta(u-t)\,u^s\,\frac{du}{u} \;=\; \begin{cases} \dfrac{2\pi}{i^s\,\Gamma(s)}\; e^{-t}\,t^{s-1} & (\text{for } t\ge 0)\\[2ex] 0 & (\text{for } t<0)\end{cases}$$

In our context this is only a *heuristic*, but it *suggests* the correct value for the integral, and we can attempt to *check* the outcome of the heuristic, via Fourier Inversion. Thus, disregarding the constant $2\pi/i^s\Gamma(s)$ for a moment, compute the inverse Fourier transform of

$$F(t) \;=\; \begin{cases} e^{-t}\,t^{s-1} & (\text{for } t\ge 0)\\[2ex] 0 & (\text{for } t<0)\end{cases}$$

This is

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi x}\,F(\xi)\,d\xi \;=\; \frac{1}{2\pi}\int_0^{\infty} e^{i\xi x}\,e^{-\xi}\,\xi^{s-1}\,d\xi \;=\; \frac{1}{2\pi}\int_0^{\infty} e^{i\xi x}\,e^{-\xi}\,\xi^s\,\frac{d\xi}{\xi}$$

$$=\;\frac{1}{2\pi}\int_0^{\infty} e^{-\xi(1-ix)}\,\xi^s\,\frac{d\xi}{\xi} \;=\; \frac{1}{2\pi}\frac{1}{(1-ix)^s}\int_0^{\infty} e^{-\xi}\,\xi^s\,\frac{d\xi}{\xi} \;=\; \frac{1}{2\pi}\frac{1}{(1-ix)^s}\Gamma(s) \;=\; \frac{1}{2\pi}\,i^s\,\frac{1}{(x+i)^s}\Gamma(s)$$

by the same identity. Thus, all the constants correctly cancel, and by Fourier Inversion we see that the heuristic gave the true outcome:

$$\int_{-\infty}^{\infty} e^{-itx}\,\frac{1}{(x+i)^s}\,dx \;=\; \begin{cases} \dfrac{2\pi}{i^s\,\Gamma(s)}\; e^{-t}\,t^{s-1} & (\text{for } t\ge 0)\\[2ex] 0 & (\text{for } t<0)\end{cases}$$

[03.6] Show that $f(z)=\displaystyle\int_0^1 \frac{dt}{t\cdot z+(1-t)\cdot z_o}$ is holomorphic at any z such that 0 is *not* on the straight line segment with endpoints z_o and z. Find the radius of convergence of its power series expanded at $z_o=-4+3i$.

As with the case $z_o=1$, holomorphy is proven via Morera's theorem, for example.

For any z_o such that the line segment connecting z_o and $-4+3i$ does not pass through 0, the corresponding function is holomorphic at $-4+3i$, so admits a power series expansion there. From Cauchy theory, this power series will converge on the largest open disk centered at $-4+3i$ on which there is a holomorphic function agreeing with $f(z)$.

Because of the potential blow-up of the integral, not to mention knowing that $\log 0$ cannot have a value making the function holomorphic, no one of these functions $f(z)$ can be holomorphic at 0, so 0 is not contained in any disk on which $f(z)$ is holomorphic. We show that there is no *other* obstacle.

The functions $f(z)$ defined via different z_o only differ by constants, the value of the integral of $1/w$ from one z_o to another. Thus, in particular, we could consider $z_o=-4+3i$ without loss of generality, in the sense that if we find radius of convergence equal to the distance to 0 (namely, 5), then, since we cannot do any *better*, we're done.

The function $f(z)$ defined with $z_o = -4+3i$ is holomorphic on the slit plane obtained by removing from $\mathbb{C}$ the ray from 0 passing through $-(-4+3i)$. The largest disk centered at $-4+3i$ in this half-plane indeed has radius 5, the distance from $-4+3i$ to 0. ///

[03.7] Show that there is a holomorphic function $f(z) = \sqrt{z^5-1}$ near any point z_o with $z_o^5 \neq 1$. Determine the radius of convergence of the power series for $f(z)$ expanded at 0.

Especially if we want a clear answer to the radius-of-convergence part of the question, it is advantageous to observe that a product of square roots of $z-\alpha$ as α runs over the zeros of z^5-1 will be a square root of the product. Thus, we analyze the individual square roots $\sqrt{z_\alpha}$ separately.

Again, on any half-plane H not containing 0, there is *a* holomorphic logarithm $L(z)$, with the property that $e^{L(z)} = z$, but we are *not* promised that $L(zw) = L(z)+L(w)$. *A* holomorphic square root $S(z)$ can be defined on H by

$$S(z) \;=\; e^{\frac{1}{2}\cdot L(z)}$$

with the property that $S(z)^2 = z$, but we are *not* promised that $S(zw) = S(z)\cdot S(w)$.

Thus, for z_o with $z_o - \alpha \neq 0$, there is a holomorphic $\sqrt{z-\alpha}$ on any half-plane containing z_o but not containing 0. In particular, as in the previous problem, the radius of convergence would be the distance from z_o to α.

Thus, near z_o with $z_o^5 - 1 \neq 0$, there is a $\sqrt{z^5-1}$, with radius of convergence at least equal to the minimum of the distances from z_o to the fifth roots of unity. With $z_o = 0$, this minimum is 1, so the radius of convergence is *at least* 1.

On the other hand, since there is a holomorphic $\sqrt{z-\alpha}$ near $z_o = 1$ for $\alpha \neq 1$, the potential obstacle to holomorphy of $\sqrt{z^5-1}$ at 1 is just the impossibility of having a holomorphic $\sqrt{z-1}$ near 1. Among other methods to show this impossibility, one is to assume that there is such a square root, expand it in a power series at 1, and obtain a contradiction: if

$$z-1 \;=\; \Big(a_o + a_1(z-1) + \ldots\Big)^2 \;=\; a_o^2 + 2a_oa_1(z-1) + \ldots$$

then $a_o = 0$, but then the $(z-1)^1$ terms cannot possibly match. Thus, there cannot be a square root of $z-1$ on any disk about 1, so the radius of convergence of the power series for (any) $\sqrt{z^5-1}$ expanded at 0 is 1. ///

[03.8] With real b, the function $f(z) = 1 + e^z + e^{bz}$ does not vanish on the real line. Estimate the number of its zeros in $|z| < R$ for large R.

Use the argument principle: the number of zeros, counting multiplicities, of holomorphic f inside a simple closed positively-oriented curve γ (not running through any zeros of f, and contractible inside an open set on which f is holomorphic) is

$$(\text{number of zeros of } 1+e^z+e^{bz} \text{ inside } \gamma) \;=\; \frac{1}{2\pi i}\int_\gamma \frac{f'(z)\,dz}{f(z)} \;=\; \frac{1}{2\pi i}\int_\gamma d\log f(z)$$

(When $b=0$ or $b=1$, the question collapses to a much simpler case of the form $e^z + c$, which we ignore.) For $b<0$, multiply through by $e^{|b|z}$ (which never vanishes!) to have $e^{|b|z} + e^{(1+|b|)z} + 1$. For $0<b<1$, replace z by z/b. Thus, we consider only $b>1$.

Let $x = \mathrm{Re}(z)$. The function cannot vanish when $|e^{bz}| > |e^z+1|$, which is implied by $e^{bx} > 2\cdot e^x$, which is implied by $x(b-1) > \log 2$, or $\mathrm{Re}(z) > \frac{\log 2}{b-1}$. That is, the function can have no zeros in that right half-plane.

A similar argument shows that there can be no zeros in $\mathrm{Re}(z) < -\log 2$. Thus, we can count the zeros in $|z| < R$ by counting zeros inside a box with vertices $\pm c \pm iR$ with $c > 0$ large enough and fixed.

In trying to estimate the total change in the argument of the logarithm around such a contour, note that, for $|e^{bz}| > A \cdot |e^z + 1|$,

$$\Big|\arg(1 + e^z + e^{bz}) \;-\; \arg(e^{bz})\Big| \;<\; \frac{1}{A}$$

Thus, integrating on $c \pm iR$ with $c > 0$ large enough so that $|e^{bz}| > A \cdot |e^z + 1|$, the total change in the argument of $f(z)$ is the total change in the argument of e^{bz} up to an error of size at most $2R/A$. That is, this total change of argument is $2R \cdot b$ up to an error at most $2R/A$.

Similarly, along $-c \pm iR$, with c sufficiently large, the total change of the argument is the total change of the argument of the constant 1, namely, 0, up to an error of size at most $2R/A$. Thus, the total change in arguments along the vertical sides is

$$\Big|(\text{change in arg along vertical sides of box}) \;-\; 2R \cdot b\Big| \;\leq\; \frac{4R}{A}$$

To estimate the change in argument along the top and bottom edges, we introduce a standard trick: first, if $\mathrm{Re}\, f(z)$ does not vanish, then the net change in $\arg f(z)$ cannot be more than π. Similarly, if $\mathrm{Re}\, f(z)$ vanishes n times, then the total change in the argument is at most $(n+1)\pi$. For $z = x + iy$,

$$\mathrm{Re}(1 + e^z + e^{bz}) \;=\; 1 + \cos y \cdot e^x + \cos by \cdot e^{bx}$$

We can choose to do this counting at $y = \mathrm{Im}(z)$ where not both $\cos y$ and $\cos by$ vanish. Between any two zeros of $\mathrm{Re}\, f(z)$, there must be a zero of $(\mathrm{Re}\, f(z))'$, and here

$$\big(\mathrm{Re}(1 + e^z + e^{bz})\big)' \;=\; \cos y \cdot e^x + b \cos by \cdot e^{bx}$$

Dividing through the latter by e^x does not affect vanishing, giving

$$\cos y + \cos by \, e^{(b-1)x}$$

Since $e^{(b-1)x}$ is *monotone*, this vanishes at most once for $x \in \mathbb{R}$. Thus, there are at most 2 zeros of $\mathrm{Re}\, f(z)$, so the argument of $f(z)$ changes by at most 3π along *any* horizontal line.

Thus, given R,

$$\Big|(\text{change in arg along all sides of box}) \;-\; 2R \cdot b\Big| \;\leq\; \frac{4R}{A} + 6\pi$$

and

$$\Big|(\text{number zeros inside}) - \frac{R \cdot b}{\pi}\Big| \;\leq\; \frac{4R}{2\pi A} + 3$$

The further important standard trick here is to exploit the fact that we can make the box *wider* without affecting the number of zeros inside, since we have seen that there are no zeros outside a certain vertical strip. Thus, we can make c large enough to have the inequality with arbitrarily large A. Thus, for example,

$$\Big|(\text{number zeros inside}) - \frac{R \cdot b}{\pi}\Big| \;\leq\; 4$$

Thus, up to an error bounded by 4, the number of zeros inside a disk of radius R is Rb/π. We can write a weaker version of this statement as

$$(\text{number zeros inside}) \;=\; \frac{R \cdot b}{\pi} + O(1)$$

where $O(1)$ denotes an error term that is bounded. ///
