(*October 23, 2014*)

Complex analysis examples 04

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/~garrett/

[This document is http://www.math.umn.edu/~garrett/m/complex/examples_2014-15/cx_discussion_04.pdf]

[04.1] Compute $\displaystyle\int_0^\infty \frac{x^s\,dx}{1+x^2}$

The integral is absolutely convergent for $-1 < \mathrm{Re}(s) < 1$. Implicitly,

$$x^s \;=\; e^{s\log x}$$

where the logarithm is the one which is real-valued on $(0,+\infty)$. Use the Hankel/keyhole contour. First, the integral itself is a limit

$$\int_0^\infty \frac{x^s\,dx}{1+x^2} \;=\; \lim_{\varepsilon\to 0^+,\; R\to+\infty} \int_\varepsilon^R \frac{x^s\,dx}{1+x^2}$$

Let $H_{\varepsilon,R}$ be the Hankel/keyhole contour that comes from R along the real line to ε, then traces a circle of radius ε around 0 counter-clockwise to ε, then back out to R. Let H_ε be the limiting case as $R\to+\infty$. We want the integral along that last part of the path, the outbound part from ε back out to R, to be the original integral $\int_\varepsilon^R x^s/(x^2+1)\,dx$. That is, we want the value of x^s to match.

On that small circle, the *argument* of x changes continuously, with a net increase of 2π from its value on the in-bound part of the path. Requiring that x^s change *continuously* on that small circle, and be $e^{s\log x}$ with real-valued $\log x$ after traversing 2π radians counter-clockwise, requires that x^s be $e^{s(\log x - 2\pi i)}$ on the in-bound path. Thus,

$$\int_{\text{outbound}+\text{inbound}} \frac{x^2\,dx}{1+x^2} \;=\; (1-e^{-2\pi i s}) \int_\varepsilon^R \frac{x^2\,dx}{1+x^2}$$

Further, the main point of the keyhole trick is that, surprisingly, the limit over $\varepsilon\to 0^+$ is *reached in finite time*, in the sense that there is sufficiently small $\varepsilon_o > 0$ such that

$$\lim_{\varepsilon\to 0^+} \int_{H_{\varepsilon,R}} \frac{x^s\,dx}{1+x^2} \;=\; \int_{H_{\varepsilon_1,R}} \frac{x^s\,dx}{1+x^2} \qquad\qquad (\text{for all positive } \varepsilon_1 < \varepsilon_o)$$

Recall the proof: for $0<\varepsilon_1<\varepsilon_o$, let $\gamma_{\varepsilon_o,\varepsilon_1}$ be the closed path that traces counter-clockwise around the circle of radius ε_o from ε_o back to ε_o, then left to ε_1, then clockwise around a circle of radius ε_1 back to ε_1, then right to ε_o. In the interior of this path, the integrand is *holomorphic*. Adding the integral over $\gamma_{\varepsilon_o,\varepsilon_1}$ to the integral over $H_{\varepsilon_1,R}$ makes the integrals from ε_o to ε_1 (inbound) and from ε_1 to ε_o (outbound) cancel, and the integrals around the circles of radius ε_1 cancel, leaving $H_{\varepsilon_o,R}$. (Yes, one should draw a picture.)

To evaluate

$$\int_{H_{\varepsilon_1,R}} \frac{x^s\,dx}{1+x^2}$$

add an integral counter-clockwise around a circle σ_R of radius R from $R\in\mathbb{R}$ back to R. For $\mathrm{Re}(s)<1$, the trivial estimate on this integral is

$$\Big|\int_{\sigma_R} \frac{x^s\,dx}{1+x^2}\Big| \;\le\; \text{length}\cdot\sup_{\sigma_R}\Big|\frac{x^s\,dx}{1+x^2}\Big| \;\le\; 2\pi R\cdot\frac{R^{\mathrm{Re}(s)}}{(R-1)^2} \;\longrightarrow\; 0 \qquad\qquad (\text{as } R\to+\infty, \text{ for } \mathrm{Re}(s)<1)$$

Thus,

$$\lim_R \int_{H_{\varepsilon_1,R}+\sigma_R} \frac{x^s\,dx}{1+x^2} \;=\; \int_{H_{\varepsilon_1}+\sigma_R} \frac{x^s\,dx}{1+x^2} \;=\; (1-e^{-2\pi i s}) \int_0^\infty \frac{x^s\,dx}{1+x^2}$$

On the other hand, the integral over the closed contour $H_{\varepsilon_1,R}+\sigma_R$ can be evaluated by *residues*: it is $-2\pi i$ times the sum of residues in its interior, since the boundary is traced *clockwise*. Inside that path, for small ε_1 and large R, there are exactly two poles, at $x=\pm i$, and both are simple. The value of $\arg x$ at $-i$ is obtained by moving *clockwise* from the $\arg x = 0$ on $(0,+\infty)$, giving $-\frac{\pi}{2}$. The argument at $+i$ is obtained by continuing *clockwise*, giving $-\frac{3\pi}{2}$. Thus,

$$\text{sum of residues} \;=\; \frac{e^{-\frac{\pi i}{2}s}}{(-i)-i} + \frac{e^{-\frac{3\pi i}{2}s}}{i-(-i)} \;=\; \frac{e^{-\frac{\pi i}{2}s}}{-2i} + \frac{e^{-\frac{3\pi i}{2}s}}{2i}$$

In summary,

$$\int_0^\infty \frac{x^s\,dx}{1+x^2} \;=\; \frac{1}{1-e^{-2\pi i s}} \lim_R \int_{H_{\varepsilon_1}+\sigma_R} \frac{x^s\,dx}{1+x^2} \;=\; \frac{-2\pi i}{1-e^{-2\pi i s}}\Big(\frac{e^{-\frac{\pi i}{2}s}}{-2i} + \frac{e^{-\frac{3\pi i}{2}s}}{2i}\Big)$$

$$=\; \frac{\pi}{1-e^{-2\pi i s}}\Big(e^{-\frac{\pi i}{2}s} - e^{-\frac{3\pi i}{2}s}\Big) \;=\; \pi\,\frac{e^{\frac{\pi i}{2}s} - e^{-\frac{\pi i}{2}s}}{e^{\pi i s}-e^{-\pi i s}} \;=\; \frac{\pi}{2}\,\frac{2}{e^{\frac{\pi i}{2}s}+e^{-\frac{\pi i}{2}s}} \;=\; \frac{\pi}{2\cos\frac{\pi s}{2}}$$

[04.2] Compute $\displaystyle\int_0^1 \frac{\big(x(1-x)\big)^s}{1+x^3}\,dx$

Oops, as it stands, I don't think that we can do much with it. Possibly what I intended, or in any case is better, was something like

$$\int_0^1 \frac{x^s\,(1-x)^{-s}}{1+x^3}\,dx \qquad\qquad (\text{with } \mathrm{Re}\,(s) > -1)$$

This does admit a variation of the Hankel/keyhole contour idea, namely, tracking $s\arg x$ counter-clockwise around 0 adds $2\pi s$, while tracking $-s\arg(1-x)$ counter-clockwise around 1 *subtracts* $2\pi s$. That is, moving around *both* $0,1$ (with the modified set-up) returns $x^s(1-x)^{-s}$ to its original value. That is, on $\mathbb{C}-[0,1]$, the complex plane with the unit interval removed, there is a well-defined holomorphic (and genuinely single-valued!) $x^s(1-x)^{-s}$.

The original integral from 0 to 1 is not cancelled by the integral from 1 back to 0 after going around 1 counter-clockwise, because $x^s(1-x)^{-s}$ has become $e^{-2\pi i x}\cdot x^s(1-x)^{-s}$. Thus, for $\varepsilon>0$, letting γ_ε be the path from ε to $1-\varepsilon$, then clockwise around 1 back to $1-\varepsilon$, then left to ε, and around 0 counter-clockwise back to ε,

$$\lim_\varepsilon \int_{\gamma_\varepsilon} \frac{x^s(1-x)^{-s}}{1+x^3}\,dx \;=\; (1-e^{-2\pi i s})\int_0^1 \frac{x^s(1-x)^{-s}}{1+x^3}\,dx$$

Let σ_R be a circle of radius R, traversed clockwise. Connect σ_R and γ_ε by suitably oriented inbound and outbound paths to create a large path τ. As usual, the inbound and outbound integrals are mutually cancelling. In the interior of τ the integrand is meromorphic, with simple poles at -1 and primitive sixth roots of 1, $\zeta = e^{\pi i/3}$, and $\zeta^{-1}=e^{-\pi i/3}$. Thus, noting that the large path is negatively oriented, so that $-2\pi i$ times the residues is picked up,

$$\int_{\gamma_\varepsilon+\sigma_R} \frac{x^s(1-x)^{-s}}{1+x^3}dx \;=\; -2\pi i\,\mathrm{Res}_{z=-1,\,\zeta,\,\zeta^{-1}}\; \frac{x^s(1-x)^{-s}}{1+x^3}$$

$$=\; -2\pi i\Big(\frac{(-1)^s(1-(-1))^{-s}}{(-1-\zeta)(-1-\zeta^{-1})} + \frac{\zeta^s(1-\zeta)^{-s}}{(1-\zeta)(\zeta-\zeta^{-1})} + \frac{(\zeta^{-1})^s(1-\zeta^{-1})^{-s}}{(1-\zeta^{-1})(\zeta^{-1}-\zeta)}\Big)$$

Then there is the task of identifying the correct s^{th} powers. Putting that off, the integral over σ_R can be easily estimated for $\mathrm{Re}\,(s)<0$ by

$$\Big|\int_{\sigma_R} \frac{x^s(1-x)^{-s}}{1+x^3}dx\Big| \;\le\; \text{length}\cdot\text{sup on path} \;\le\; 2\pi R\cdot\frac{(R+1)^{2\mathrm{Re}\,(s)}}{(R-1)^3} \;\longrightarrow\; 0 \qquad\qquad (\text{as } R\to\infty)$$

Thus, with suitable values of s^{th} powers,

$$\int_0^1 \frac{x^s(1-x)^{-s}}{1+x^3}\,dx \;=\; \frac{-2\pi i}{1-e^{-2\pi i s}}\cdot\Big(\frac{\big(\frac{-1}{1-(-1)}\big)^s}{(-1-\zeta)(-1-\zeta^{-1})}+\frac{\big(\frac{\zeta}{1-\zeta}\big)^s}{(\zeta-(-1))(\zeta-\zeta^{-1})}+\frac{\big(\frac{\zeta^{-1}}{1-\zeta^{-1}}\big)^s}{(\zeta^{-1}-(-1))(\zeta^{-1}-\zeta)}\Big)$$

Last, tracking args. Since $(x/(1-x))^s$ is well-defined on $\mathbb{C}-[0,1]$, it shouldn't make any difference how we do this, as long as we consider $x/(1-x)$ as a single entity. Going from $[0,1]$ clockwise around 0 to -1 decreases the argument of $\frac{x}{1-x}$ from 0 to $-\pi$. Thus,

$$\Big(\frac{-1}{1-(-1)}\Big)^s \;=\; \Big(-\frac{1}{2}\Big)^s \;=\; e^{s(-\log 2-\pi i)} \;=\; 2^{-s}e^{\pi i s}$$

From $[0,1]$ clockwise to ζ decreases the argument of $\frac{x}{1-x}$ from 0 to

$$\arg\Big(\frac{\zeta}{1-\zeta}\Big) \;=\; \arg\Big(\frac{\zeta}{\zeta^{-1}}\Big) \;=\; \arg\zeta^2 \;=\; -\frac{4}{3}\pi$$

Thus,

$$\Big(\frac{\zeta}{1-\zeta}\Big)^s \;=\; (\zeta^2)^s \;=\; e^{s(-\frac{4}{3}\pi)}$$

From $[0,1]$ clockwise to ζ^{-1} decreases the argument of $\frac{x}{1-x}$ from 0 to

$$\arg\Big(\frac{\zeta^{-1}}{1-\zeta^{-1}}\Big) \;=\; \arg\Big(\frac{\zeta^{-1}}{\zeta}\Big) \;=\; \arg\zeta^{-2} \;=\; -\frac{2}{3}\pi$$

Thus,

$$\Big(\frac{\zeta^{-1}}{1-\zeta^{-1}}\Big)^s \;=\; (\zeta^{-2})^s \;=\; e^{s(-\frac{2}{3}\pi)}$$

Thus,

$$\int_0^1 \frac{x^s(1-x)^{-s}}{1+x^3}\,dx \;=\; \frac{-2\pi i}{1-e^{-2\pi i s}}\Big(\frac{2^{-s}e^{\pi i s}}{(1+\zeta)(1+\zeta^{-1})}+\frac{e^{-\frac{4}{3}\pi i s}}{(\zeta+1)(i\sqrt{3})}+\frac{e^{-\frac{2}{3}\pi i s}}{(\zeta^{-1}+1)(-i\sqrt{3})}\Big)$$

Perhaps further simplification is of less interest... although one might hope to certify that for $s\in\mathbb{R}$ this apparent outcome is real. ///

[04.3] Compute $\displaystyle\int_0^\infty e^{-i\xi x}\,x^s\,e^{-x}\,dx$ with $\mathrm{Re}(s)>-1$.

This invites application of the Gamma identity

$$\int_0^\infty e^{-xy}\,x^s\,\frac{dx}{x} \;=\; y^{-s}\int_0^\infty e^{-x}\,x^s\,\frac{dx}{x} \;=\; y^{-s}\,\Gamma(s)$$

which holds first for $y>0$ and then for $\mathrm{Re}(y)>0$, by the Identity Principle (also known as *The Permanence of Analytic Relationships*):

$$\int_0^\infty e^{-i\xi x}\,x^s\,e^{-x}\,dx \;=\; \int_0^\infty e^{-x(1+i\xi x)}\,x^{s+1}\,e^{-x}\,\frac{dx}{x}$$

$$=\;(1+i\xi)^{-(s+1)}\int_0^\infty e^{-x}\,x^{s+1}\,e^{-x}\,\frac{dx}{x} \;=\; (1+i\xi)^{-(s+1)}\cdot\Gamma(s+1)$$

[04.4] Compute $\int_{-\infty}^{\infty} e^{-i\xi x}\, xe^{-x^2}\, dx$

This is the Fourier transform of $x \to xe^{-x^2}$. We can reduce it to a slightly simpler computation by an integration by parts:

$$\int_{-\infty}^{\infty} e^{-i\xi x}\, xe^{-x^2}\, dx \;=\; -\tfrac{1}{2}\int_{-\infty}^{\infty} e^{-i\xi x}\, \frac{d}{dx}e^{-x^2}\, dx \;=\; \tfrac{1}{2}\int_{-\infty}^{\infty} \frac{d}{dx}e^{-i\xi x}\cdot e^{-x^2}\, dx \;=\; -\tfrac{1}{2}i\xi\int_{-\infty}^{\infty} e^{-i\xi x}\, e^{-x^2}\, dx$$

The exponentials can be combined, and then complete the square:

$$\int_{-\infty}^{\infty} e^{-i\xi x}\, e^{-x^2}\, dx \;=\; \int_{-\infty}^{\infty} e^{-(x^2+i\xi x)}\, dx \;=\; \int_{-\infty}^{\infty} e^{-(x^2+i\xi x-\frac{\xi^2}{4})-\frac{\xi^2}{4}}\, dx \;=\; e^{-\frac{\xi^2}{4}}\int_{-\infty}^{\infty} e^{-(x+\frac{i\xi}{2})^2}\, dx$$

The intuition at this point is that sliding the integral from $-\infty$ to $+\infty$ along the real axis to be an integral from $-i\xi-\infty$ to $-i\xi+\infty$ will not change the value of the integral, since there are no residues to pick up, while it will convert the integrand back to e^{-x^2}, which does not involve ξ.

As usual, an integral from $-\infty$ to $+\infty$ is a limit of the corresponding integral from $-R$ to $+R$, as $R \to +\infty$. Then

$$\int_{-\infty}^{\infty} e^{-(x+\frac{i\xi}{2})^2}\, dx \;=\; \lim_R \int_{-R}^{R} e^{-(x+\frac{i\xi}{2})^2}\, dx \;=\; \int_{-i\xi-R}^{-i\xi+R} e^{-x^2}\, dx$$

Let B_R be the rectangle with vertices $\pm R$ and $-i\xi \pm R$, traced counter-clockwise. The integrals over the ends of the box are easily estimated: since $|e^{-(x+iy)^2}| = e^{-\mathrm{Re}\,(x+iy)^2} = e^{-x^2+y^2}$,

$$\Big|\int_{R}^{-i\xi+R} e^{-x^2}\, dx\Big| \;\le\; \text{length}\cdot(\text{sup on curve}) \;\le\; |\xi|\cdot e^{-R^2}\cdot e^{\xi^2} \;\longrightarrow\; 0 \qquad\qquad (\text{as } R\to+\infty)$$

Thus,

$$0 \;=\; \lim_{R\to\infty} 0 \;=\; \lim_R \int_{B_R} e^{-i\xi x}\, e^{-x^2}\, dx \;=\; \lim_R \Big(e^{-\frac{\xi^2}{4}}\int_{-i\xi-R}^{-i\xi+R} e^{-x^2}\, dx - e^{-\frac{\xi^2}{4}}\int_{-R}^{R} e^{-x^2}\, dx\Big)$$

so

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx \;=\; e^{-\frac{\xi^2}{4}}\cdot\int_{-\infty}^{\infty} e^{-x^2}\, dx \;=\; e^{-\frac{\xi^2}{4}}\cdot\sqrt{\pi}$$

and

$$\int_{-\infty}^{\infty} e^{-i\xi x}\, xe^{-x^2}\, dx \;=\; -\tfrac{1}{2}i\xi\int_{-\infty}^{\infty} e^{-i\xi x}\, e^{-x^2}\, dx \;=\; -\tfrac{1}{2}i\xi\cdot e^{-\frac{\xi^2}{4}}\cdot\sqrt{\pi}$$

[04.5] For continuous φ on the unit circle $|z| = 1$, define

$$f_\varphi(z) \;=\; \int_0^{2\pi} \frac{\varphi(e^{i\theta})}{e^{i\theta}-z}\, d\theta \qquad\qquad (\text{for } |z| < 1)$$

Show that $f(z)$ is holomorphic. Give an example of φ *not* identically 0 so that f_φ *is* identically 0.

Use Morera's theorem: with γ be a small counter-clockwise triangle around a given z_o in the open unit disk,

$$\int_\gamma f_\varphi(z)\, dz \;=\; \int_\gamma\int_0^{2\pi} \frac{\varphi(e^{i\theta})}{e^{i\theta}-z}\, d\theta\, dz \;=\; \int_0^{2\pi} \varphi(e^{i\theta})\Big(\int_\gamma \frac{dz}{e^{i\theta}-z}\Big)d\theta \;=\; \int_0^{2\pi} 0\, d\theta \;=\; 0$$

Morera's theorem says that this vanishing implies holomorphy of f_φ. ///

Note that the given integral is not quite a written-out version of Cauchy's kernel, because $d(e^{i\theta}) = i\theta\, e^{i\theta}\, d\theta$, so a factor of $e^{i\theta}$ is missing. Nevertheless, it's *close*. Thus, various heuristics might suggest making $\varphi(e^{i\theta})$ be the boundary value of an *anti-holomorphic* function such as $F(z) = \overline{z}$. Thus, $\varphi(e^{i\theta}) = F(e^{i\theta}) = e^{-i\theta}$. For $|z| < 1$, expanding a geometric series:

$$f_\varphi(z) \;=\; \int_0^{2\pi} \frac{\varphi(e^{i\theta})}{e^{i\theta} - z}\, d\theta \;=\; \int_0^{2\pi} \frac{e^{-i\theta}}{e^{i\theta} - z}\, d\theta \;=\; \int_0^{2\pi} e^{-i\theta}\, \frac{e^{-i\theta}}{1 - ze^{-i\theta}}\, d\theta \;=\; \sum_{n=0}^\infty \int_0^{2\pi} e^{-2i\theta} \big(ze^{-i\theta}\big)^n\, d\theta$$

$$= \sum_{n=0}^\infty z^n \int_0^{2\pi} e^{-i(2+n)\theta}\, d\theta \;=\; \sum_{n=0}^\infty z^n \cdot 0 \;=\; 0$$

Thus, with hindsight, $\varphi(e^{i\theta}) = 1$ would also have given $f_\varphi = 0$. ///

[04.6] Let f be an entire function such that $f(z+1) = f(z)$ and $f(z+i) = f(z)$ for all z. Show that f is constant.

First, the given *periodicity relations* imply that all the values of f are determined by its values on $R = \{z = x+iy : 0 \le x \le 1,\ 0 \le y \le 1\}$: given x, y, there are unique integers m, n such that $m \le x < m+1$ and $n \le y < n+1$. By the obvious induction,

$$f(x+iy) \;=\; f((x-m) + i(y-n))$$

while $0 \le x - m < 1$ and $0 \le y - n < 1$. On the compact set $0 \le x \le 1$ and $0 \le y \le 1$, the continuous function f is *bounded*. Thus, f is entire and bounded, so by Liouville, it is constant. ///

[04.7] Show that a *real-valued* holomorphic function is constant.

For f real-valued on a neighborhood of z_o, taking a derivative along a *real* direction, but also along a *purely imaginary* direction, gives

$$f'(z_o) \;=\; \lim_{\varepsilon \to 0} \frac{f(z_o + \varepsilon) - f(z_o)}{\varepsilon} \;=\; \lim_{\varepsilon \to 0} \frac{f(z_o + i\varepsilon) - f(z_o)}{i\varepsilon} \qquad \text{(with } \varepsilon \text{ real)}$$

The first limit is real, the second imaginary, so the equality implies that they are both 0. Thus, $f' = 0$, and f is constant. ///

Another kind of argument, applicable to *entire* functions with constrained values: for f were entire and real-valued, the function $F(z) = e^{if(z)}$ takes values on the unit circle. In particular, F is *bounded* and entire, so *constant*, by Liouville. Then $0 = F'(z) = if'(z)e^{if(z)}$, so $f'(z) = 0$, and f is constant. ///

[04.8] The *Bergmann kernel* of the unit disk is

$$K(z, w) \;=\; \frac{1}{\pi}\, \frac{1}{(1 - \overline{w}\, z)^2}$$

For f holomorphic on the open unit disk and extending continuously to a continuous function on the *closed* unit disk, show that

$$f(w) \;=\; \int\!\!\int_{x^2+y^2 \le 1} f(x+iy)\, \overline{K(z, w)}\, dx\, dy$$

In fact, it is better to *derive* the kernel from first principles. That is, holomorphic functions on the unit disk that extend to be continuous on the closed disk are *bounded*, so we can put the *hermitian inner product*

$$\langle f, g \rangle \;=\; \int\!\!\int_{x^2+y^2 \le 1} f(x+iy)\, \overline{g(x+iy)}\, dx\, dy$$

on the $\mathbb{C}$-vectorspace of such functions. It is natural to wonder about $\langle z^m, z^n \rangle$:

$$\langle z^m, z^n \rangle \;=\; \int\int_{x^2+y^2\le 1} z^m\, \overline{z}^n \, dx\, dy \;=\; \int_0^1 \int_0^{2\pi} r^{m+n}\, e^{\pi i(n-m)\theta}\, d\theta\, r\, dr$$

$$= \delta_{mn}\, 2\pi \int_0^1 r^{2n}\, r\, dr \;=\; \delta_{mn} \frac{\pi}{n+1}$$

with $\delta_{mn} = 1$ if $m = n$ and 0 otherwise. Thus, $u_n(z) = z^n \cdot \sqrt{\frac{n+1}{\pi}}$ is an orthonormal basis, and the *reproducing kernel*, or *Bergmann kernel*, is

$$K(z,w) \;=\; \sum_n u_n(z)\cdot \overline{u_n(w)} \;=\; \frac{1}{\pi}\sum_{n=0}^{\infty}(n+1)\, z^n\, \overline{w}^n \;=\; \frac{1}{\pi}\sum_{n=0}^{\infty} \frac{1}{\overline{w}} \frac{d}{dz}(z\overline{w})^{n+1}$$

$$= \frac{1}{\pi}\frac{1}{\overline{w}}\frac{d}{dz}\frac{z\overline{w}}{1-z\overline{w}} \;=\; \frac{1}{\pi}\frac{1}{(1-z\overline{w})^2}$$

Just to check, use the power series expansion $f(z) = \sum_{n\ge 0} c_n\, z^n$, expand the kernel as a geometric series

$$\frac{1}{\pi}\,\frac{1}{(1-\overline{w}\,z)^2} \;=\; \frac{1}{\pi}\,\frac{1}{\overline{w}}\frac{d}{dz}\frac{1}{1-\overline{w}\,z} \;=\; \frac{1}{\pi}\,\frac{1}{\overline{w}}\frac{d}{dz}\Big(1+\overline{w}\,z+(\overline{w}\,z)^2+\ldots\Big) \;=\; \frac{1}{\pi}\,\Big(1+2\overline{w}\,z+3(\overline{w}\,z)^2+\ldots\Big)$$

Then the integral is

$$\frac{1}{\pi}\sum_{m\ge 0,\; n\ge 0} c_n \int\int_{x^2+y^2\le 1} z^n\, (m+1)w^m\, \overline{z}^m\, dx\, dy$$

In polar coordinates $z = re^{i\theta}$, this becomes

$$\frac{1}{\pi}\sum_{m\ge 0,\; n\ge 0} c_n \int_0^1 \int_0^{2\pi} r^{m+n}\, (m+1)w^m\, e^{i(n-m)\theta}\, d\theta\, r\, dr \;=\; \frac{1}{\pi}\sum_{n\ge 0} c_n\, 2\pi(n+1)w^n \int_0^1 r^{2n}\, r\, dr$$

$$= \sum_{n\ge 0} c_n\, 2(n+1)w^n \frac{1}{2n+2} \;=\; \sum_{n\ge 0} c_n\, w^n \;=\; f(w)$$
