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Complex analysis examples discussion 10

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/~garrett/

[This document is
http://www.math.umn.edu/~garrett/m/complex/examples_2014-15/cx_discussion_10.pdf]

If you want feedback from me on your treatment of these examples, please get your work to me by Friday, Mar 27, preferably as a PDF emailed to me.

[10.1] Show that there is a well-defined, holomorphic function $1/\sqrt{1+z^4}$ on the region $|z|>1$. Show that $\int_\gamma \frac{dz}{\sqrt{1+z^4}} = 0$, where γ traces out $|z|=2$.

There is a function $1/\sqrt{1+z^4 \pm i}$ well-defined on $|z|>1$, because

$$\frac{1}{\sqrt{1+z^4}} \;=\; \frac{1}{\sqrt{z^4(1+\frac{1}{z^4})}} \;=\; \frac{1}{z^2}\cdot\frac{1}{\sqrt{1+\frac{1}{z^4}}}$$

For $|1/z^4|<1$, the quantity $1+\frac{1}{z^4}$ stays in the right half-plane, so has a holomorphic square root throughout $|z|>1$. The Laurent expansion of the original function is then

$$\frac{1}{\sqrt{1+z^4}} \;=\; \frac{1}{z^2}\cdot\frac{1}{\sqrt{1+\frac{1}{z^4}}} \;=\; \frac{1}{z^2}\cdot(1+\frac{1}{z^4})^{-\frac{1}{2}} \;=\; \frac{1}{z^2}\cdot\Big(1-\tfrac{1}{2}\frac{1}{z^4}+\ldots\Big)$$

By Cauchy's theorem, the path integral of z^n around $|z|=2$ is 0 except for $n=-1$, in which case it is $2\pi i$. But there is no $1/z$ term in that Laurent expansion. ///

[10.2] Let γ be a simple closed path counter-clockwise encircling $0,2$, and not enclosing -2. Let δ be a simple closed path counter-clockwise encircling $-2,0$, and not enclosing 2. Show that there is a holomorphic function $1/\sqrt{z(z^2-4)}$ on the annulus $1<|z-1|<3$, and a holomorphic function $1/\sqrt{z(z^2-4)}$ on the annulus $1<|z+1|<3$. Show that the two *periods*

$$\int_\gamma \frac{dz}{\sqrt{z(z^2-4)}} \qquad\qquad \int_\delta \frac{dz}{\sqrt{z(z^2-4)}}$$

are *linearly independent* over $\mathbb{R}$.

In fact, one is purely imaginary and the other is purely real. To show holomorphy in $1<|z-1|<3$ and evaluate the integral around γ, we determine (to some degree!) a Laurent expansion in that annulus. First,

$$z(z^2-4) \;=\; ((z-1)+1)((z-1)-1)((z-1)+3) \;=\; 3(z-1)^2\cdot\Big(1+\frac{1}{z-1}\Big)\Big(1-\frac{1}{z-1}\Big)\Big(1+\frac{z-1}{3}\Big)$$

Thus, the square root of the reciprocal is

$$\frac{1}{\sqrt{3}\,(z-1)}\cdot\Big(1+\frac{1}{z-1}\Big)^{-\frac{1}{2}}\Big(1-\frac{1}{z-1}\Big)^{-\frac{1}{2}}\Big(1+\frac{z-1}{3}\Big)^{-\frac{1}{2}}$$

and although we cannot easily determine the coefficient of $(z-1)^{-1}$ in elementary terms, it is *real*, so the integral gives $2\pi i$ times a real number.

Similarly, to obtain a Laurent expansion in the annulus $1<|z+1|<3$,

$$z(z^2-4) \;=\; ((z+1)-1)((z+1)+1)((z+1)-3) \;=\; -3(z+1)^2\cdot\Big(1-\frac{1}{z+1}\Big)\Big(1+\frac{1}{z+1}\Big)\Big(1-\frac{z+1}{3}\Big)$$

Note the sign! Taking a square root will give a Laurent expansion with *purely imaginary* coefficients, so the integral gives $2\pi i$ times a purely imaginary number, thus, a real number.

How to check that these integrals are non-zero? Keeping in mind that we do not expect to be able to evaluate them in more elementary terms, nevertheless we can hope to convert them to forms which are non-vanishing for essentially elementary reasons. One approach is to deform the given contours to be *Hankel/keyhole* contours, as follows. Because the denominator is essentially of order $R^{3/2}$ for large $R = |z+1|$, and of order $r^{\frac{1}{2}}$ for small $r = |z+1|$, the path enclosing $-2, 0$ (and not enclosing $+2$) can be deformed to an integral along a keyhole contour H^ε from $+2$ to $+\infty$ with a small circle of radius $\varepsilon > 0$ about 2. Recall that for sufficiently small $\varepsilon > 0$ the value of the integral is independent of ε. To match the outcome of the Laurent expansion, the integrand $1/\sqrt{z(z^2-4)}$ is required to take a purely imaginary value when the path crosses the real interval $(1, 2)$ at $2-\varepsilon$. Thus, for continuity, the integrand is *real* on $(2, +\infty)$, and without loss of generality *non-negative*. The integral over the Hankel contour is

$$\int_{H_\varepsilon} \frac{dz}{\sqrt{z(z^2-4)}} \;=\; \frac{1}{1-e^{\pi i}} \int_2^\infty \frac{dt}{\sqrt{t(t^2-4)}} \;=\; \tfrac{1}{2} \cdot \int_2^\infty \frac{dt}{\sqrt{t(t^2-4)}} \;>\; 0$$

A similar argument applies to prove that the other *period* is non-zero. Thus, since one is purely imaginary and the other purely real, they are linearly independent over $\mathbb{R}$. ///

[10.3] Show that for irrational $\alpha \in \mathbb{R}$, the set $\{m + n\alpha : m, n \in \mathbb{Z}\}$ is *dense* in $\mathbb{R}$.

(Kronecker) Let Γ be the topological closure of $G = \mathbb{Z} + \mathbb{Z}\alpha$ in $\mathbb{R}$. Suppose for a moment that we know the classification of all topologically-closed subgroups of $\mathbb{R}$: either $\{0\}$, $\mathbb{R}$, or of the form $\mathbb{Z} \cdot \beta$ for some $\beta \in \mathbb{R}$. The first case cannot occur for G. If the last case occurs, then there are integers k, ℓ such that $k \cdot \beta = 1$ and $\ell \cdot \beta = \alpha$. But then $\alpha = \ell/k \in \mathbb{Q}$, contradiction.

To prove the classification, for $\Gamma \neq \{0\}$, closed under additive inverses, Γ contains *positive* elements. In the case that there is a *least* positive element μ, claim that $\Gamma = \mathbb{Z} \cdot \mu$. Indeed, for $\gamma \in \Gamma$, by the archimedean property of $\mathbb{R}$ there is $n \in \mathbb{Z}$ such that $n\mu \leq \gamma < (n+1)\mu$. Necessarily $n\mu = \gamma$, or else $0 < \gamma - n\mu < \mu$, contradicting the minimality.

In the case that there is *not* least positive μ, let $\mu_1 > \mu_2 > \ldots > 0$ be an infinite descending sequence of positive elements of Γ. The inf γ_o is in Γ, since Γ is topologically closed. Replace μ_n by $\mu_n - \gamma_o$ to be able to assume that $\mu_n \to 0$. Again by archimedean-ness, $\mathbb{Z} \cdot \mu_n$ contains elements within μ_n of every real number. Since $\mu_n \to 0$, for every $\varepsilon > 0$ Γ contains elements within ε of every real number. By closed-ness, $\Gamma = \mathbb{R}$. ///

[10.4] Let $v_1, \ldots, v_n$ be linearly independent vectors in $\mathbb{R}^n$, and $\Lambda = \mathbb{Z}v_1 + \ldots + \mathbb{Z}v_n$ the lattice generated by them. Let $\mathbb{R}^n$ have its usual inner product and associated metric. For $r > 0$ let B_r be the ball of radius 0 centered at $0 \in \mathbb{R}^n$. Show that for small-enough $r > 0$ we have $B_r \cap \Lambda = \{0\}$.

Let A be the invertible n-by-n real matrix so that $Av_i = e_i$, where $\{e_i\}$ is the standard basis of $\mathbb{R}^n$. The map $A : \mathbb{R}^n \to \mathbb{R}^n$ by $v \to Av$ is continuous, and has continuous inverse given by multiplication by A^{-1}. Thus, there is a small-enough neighborhood U of $0 \in \mathbb{R}^n$ such that the image AU is contained in the ball $B_{\frac{1}{2}}$ of radius $\frac{1}{2}$ centered at 0. Certainly $B_{\frac{1}{2}} \cap \mathbb{Z}^n = \{0\}$. Then

$$U \cap \Lambda \;\subset\; A^{-1}\big(B_{\frac{1}{2}} \cap \mathbb{Z}^n\big) \;=\; A^{-1}\{0\} \;=\; \{0\}$$

Certainly U contains some ball at 0. ///

[10.5] Let Λ be a *lattice* in $\mathbb{R}^n$, that is, the $\mathbb{Z}$-module generated by n vectors linearly independent over $\mathbb{R}$. Prove that

$$\sum_{0 \neq \lambda \in \Lambda} \frac{1}{|\lambda|^s}$$

is absolutely convergent for $\mathrm{Re}(s) > n$, where $|\cdot|$ is the usual *length* in $\mathbb{R}^n$. (Do not invoke any non-existent *integral tests* in several variables, despite the fact that the idea of such gives a good heuristic.)

Use *volume* to give a reasonable upper bound on the number of lattice points in *shells* $\ell \le |x| \le \ell+1$. Namely, there is $0 < r < 1$ so that balls of radius r centered at lattice points do not touch each other. If a lattice point λ is in the shell $\ell \le |x| \le \ell+1$, then the ball or radius r around is is inside the thickened shell $\ell - r \le |x| \le \ell + r$, which is inside the shell $\ell-1 \le |x| \le \ell+1$. The latter has volume $C \cdot ((\ell+1)^n - (\ell-1)^n$ for a constant C depending on n. The total volume of all the balls of radius r around lattice points is at most the volume of the thickened shell. Thus, the number of lattice points inside that shell has a good upper bound:

$$\#\text{ lattice points in } \{x \in \mathbb{R}^n : \ell \le |x \le \ell+1\} \;\le\; \frac{C \cdot ((\ell+1)^n - (\ell-1)^n}{C \cdot r^n} \;\le\; C' \cdot \ell^{n-1} \qquad (\text{for } \ell > 1)$$

for some constant C'. Similarly, for any radius R, the sum over $|\lambda| < R$ is *finite*: for such lattice points, the balls of radius r around them are disjoint, and all lie inside the sphere of radius $R + r$ at 0, which has finite volume. Thus, to prove convergence, we can drop all $\lambda \in \Lambda$ with $|\lambda| \le R$. Thus, for real $s > 0$ and $R = 3$, we can bound the lattice-point sum by a one-dimensional sum:

$$\sum_{\lambda \in \Lambda,\, |\lambda| \ge 3} \frac{1}{|\lambda|^s} = \sum_{\ell=3}^\infty \sum_{\ell-1 \le |\lambda| < \ell+1} \frac{1}{|\lambda|^s} \le \sum_{\ell=3}^\infty \sum_{\ell-1 \le |\lambda| < \ell+1} \frac{1}{(\ell-1)^s}$$

$$\le C' \cdot \sum_{\ell=3}^\infty \ell^{n-1} \frac{1}{(\ell-1)^s} = C' \cdot \sum_{\ell=2}^\infty (\ell+1)^{n-1} \frac{1}{\ell^s} \le C' \cdot 2^{n-1} \sum_{\ell=2}^\infty \ell^{n-1} \frac{1}{\ell^s}$$

The usual one-dimensional integral test gives convergence of the latter for $s - (n-1) > 1$, that is, for $s > n$. ///

[10.6] Recall that we need *finite growth order* $|f(z)| \ll e^{|z|^N}$ as $|z| \to +\infty$ in a strip $a \le \mathrm{Re}(z) \le b$, for *some* N, before we can invoke the Phragmén-Lindelöf theorem. Use the integral representation of $\zeta(s)$ via $\theta(y)$, and properties of $\Gamma(s)$, to show that it has finite order of growth in $-1 \le \mathrm{Re}(s) \le 2$.

The integral representation

$$\pi^{-\frac{s}{2}} \Gamma(\tfrac{s}{2}) \cdot \zeta(s) \;=\; \int_1^\infty (y^{s/2} + y^{\frac{1-s}{2}}) \frac{\theta(y)-1}{2} \frac{dy}{y} + \frac{1}{s-1} - \frac{1}{s} \qquad (\text{with } \theta(y) = \textstyle\sum_{n \in \mathbb{Z}} e^{-\pi n^2 y})$$

converges absolutely for s away from $0, 1$. Thus, we can present $s(1-s) \cdot \zeta(s)$ in a way that makes sense for *all* $s \in \mathbb{C}$:

$$s(1-s) \cdot \zeta(s) \;=\; s(1-s) \cdot \frac{\pi^{\frac{s}{2}}}{\Gamma(\frac{s}{2})} \cdot \Big(\int_1^\infty (y^{s/2} + y^{\frac{1-s}{2}}) \frac{\theta(y)-1}{2} \frac{dy}{y} + \frac{1}{s-1} - \frac{1}{s} \Big)$$

$$= \frac{\pi^{\frac{s}{2}}}{\Gamma(\frac{s}{2})} \cdot \Big(s(1-s) \int_1^\infty (y^{s/2} + y^{\frac{1-s}{2}}) \frac{\theta(y)-1}{2} \frac{dy}{y} - s - (1-s) \Big)$$

The function $s \to \pi^{s/2}$ visibly has growth-order 1. Stirling's asymptotic shows that in $\mathrm{Re}(s) \ge \frac{1}{2}$ the function $s \to 1/\Gamma(s)$ has growth order 1, and the reflection relation $1/\Gamma(1-s) = \frac{\sin \pi s}{\pi \Gamma(s)}$ yields the growth-order estimate in $\mathrm{Re}(s) \le \frac{1}{2}$. The integral is *bounded* in vertical strips of finite width, and the polynomials s and $s-1$ have growth order 0. Since the product and/or sum of functions of growth order $\alpha > 0$ is again such a function, this proves that $s(1-s) \cdot \zeta(s)$ has growth order 1. (Phragmén-Lindelöf then gives a sharper assertion.) ///

[10.7] Show that $f(x,y) = (x \pm iy)^\ell \, e^{-\pi(x^2+y^2)}$ is multiplied by $i^{-\ell}$ by Fourier transform

$$\widehat{f}(\xi, \eta) \;=\; \int_{\mathbb{R}^2} e^{-2\pi i(\xi x + \eta y)} \, f(x,y) \, dx \, dy$$

Hint: rewrite this in terms of $z = x + iy$ and $\overline{z}$, and another complex variable $w = \xi + i\eta$ and $\overline{w}$, and look for a chance to differentiate under the integral defining the Fourier transform.

Following the hint, and taking the plus sign, the Fourier transform is

$$\widehat{f}(w,\overline{w}) \;=\; \int_{\mathbb{R}^2} e^{-\pi i(z\overline{w}+\overline{z}w)}\; z^\ell\, e^{-\pi z\overline{z}}\;dx\,dy \;=\; \frac{1}{(-\pi i)^\ell}\int_{\mathbb{R}^2}\Big(\frac{\partial}{\partial\overline{w}}\Big)^\ell e^{-\pi i(z\overline{w}+\overline{z}w)}\; e^{-\pi z\overline{z}}\;dx\,dy$$

$$=\; \frac{1}{(-\pi i)^\ell}\cdot\Big(\frac{\partial}{\partial\overline{w}}\Big)^\ell\int_{\mathbb{R}^2} e^{-\pi i(z\overline{w}+\overline{z}w)}\; e^{-\pi z\overline{z}}\;dx\,dy \;=\; \frac{1}{(-\pi i)^\ell}\Big(\frac{\partial}{\partial\overline{w}}\Big)^\ell e^{-\pi w\overline{w}}$$

since we already know that these pure Gaussians map to themselves under Fourier transform. Then this is

$$\frac{1}{(-\pi i)^\ell}(-\pi w)^\ell\cdot e^{-\pi w\overline{w}} \;=\; i^{-\ell}\cdot w^\ell\, e^{-\pi w\overline{w}}$$

as claimed. The argument for the minus-sign case is identical. ///

[10.8] Define a *harmonic theta function* $\Theta_\ell(y)$ by

$$\Theta_\ell(y) \;=\; \begin{cases} \dfrac{1}{4}\displaystyle\sum_{(0,0)\neq(m,n)\in\mathbb{Z}^2} (m+in)^\ell\, e^{-\pi y(m^2+n^2)} & (\text{for } \ell>0)\\ \dfrac{1}{4}\displaystyle\sum_{(0,0)\neq(m,n)\in\mathbb{Z}^2} (m-in)^{|\ell|}\, e^{-\pi y(m^2+n^2)} & (\text{for } \ell<0)\end{cases}$$

Show that this is identically 0 unless ℓ is divisible by 4, and prove the *functional equation*

$$\Theta_\ell(1/y) \;=\; y^{\ell+1}\cdot\Theta_\ell(y)$$

Replacing $m\pm in$ by $i\cdot(m\pm in)$ is a bijection of non-zero Gaussian integers to themselves, so cannot alter the sum. Yet a factor of i^ℓ comes out, so the whole sum is multiplied by i^ℓ. Thus, either the sum is 0, or $i^\ell = 1$.

In $\mathbb{R}^2$, by changing variables in the defining integral, $f(\sqrt{y}\cdot v)\widehat{\;} = \frac{1}{y}\widehat{f}(\frac{1}{y}\cdot v)$. Let $f(u,v) = (u\pm iv)\, e^{-\pi(u^2+v^2)}$. Note that the term $m=n=0$ in the sum defining the theta function vanishes for $\ell\neq 0$, and we consider only that situation. Also, take $\ell\in 4\mathbb{Z}$. The previous example's computation of the Fourier transform of $(u\pm iv)\, e^{-\pi(u^2+v^2)}$, Poisson summation, give

$$\Theta(y) \;=\; \sum_{m,n\in\mathbb{Z}} (m\pm in)^\ell\, e^{-\pi y(m^2+n^2)} \;=\; y^{-\ell/2}\sum_{m,n\in\mathbb{Z}} (\sqrt{y}(m\pm in))^\ell\, e^{-\pi y(m^2+n^2)} \;=\; y^{-\ell/2}\sum_{m,n\in\mathbb{Z}} f(\sqrt{y}\cdot(m,n))$$

$$=\; y^{-\ell/2}\frac{1}{y}\sum_{m,n\in\mathbb{Z}}\widehat{f}(\frac{1}{\sqrt{y}}\cdot(m,n)) \;=\; y^{-\ell/2}\frac{1}{y}i^{-\ell}\sum_{m,n\in\mathbb{Z}}(\frac{1}{\sqrt{y}}(m\pm in))^\ell\, e^{-\pi y(m^2+n^2)}$$

$$=\; y^{-\ell}\frac{1}{y}\sum_{m,n\in\mathbb{Z}}(m\pm in)^\ell\, e^{-\pi y(m^2+n^2)} \;=\; \frac{1}{y^{\ell+1}}\Theta(\frac{1}{y})$$

as claimed. ///

[10.9] Let $\chi(\alpha) = (\alpha/|\alpha|)^\ell$ for $\alpha\in\mathbb{C}^\times$. The associated *Hecke L-function* on the Gaussian integers $\mathbb{Z}[i]$ is

$$L(s,\chi) \;=\; \frac{1}{\#\mathbb{Z}[i]^\times}\sum_{0\neq\alpha\in\mathbb{Z}[i]}\frac{\chi(\alpha)}{|\alpha|^{2s}} \;=\; \frac{1}{4}\sum_{0\neq\alpha\in\mathbb{Z}[i]}\frac{\chi(\alpha)}{|\alpha|^{2s}}$$

Show that this is identically 0 unless ℓ is divisible by 4. Prove that $L(s,\chi_\ell)$ has an analytic continuation and functional equation and has the integral representation

$$\pi^{-(s+\frac{|\ell|}{2})}\,\Gamma\Big(s+\frac{|\ell|}{2}\Big)\,L(s,\chi) \;=\; \int_0^\infty y^{s+\frac{|\ell|}{2}}\,\Theta_\ell(y)\,\frac{dy}{y} \qquad\qquad (\text{for } \mathrm{Re}(s)>1)$$

As with the theta functions in the previous, the change of variables by multiplying $m\pm in$ by i merely permutes the summands, but multiplies the whole sum by i^ℓ, so either the sum is 0 or $i^\ell =$. So take $\ell\in 4\mathbb{Z}$. Also, the case $\ell=0$ was treated earlier, so take $\ell\neq 0$.

As with Riemann's zeta and other examples, these L-functions are Mellin transforms of the theta functions, once the normalizations are correctly determined. In the case $\ell>0$,

$$\int_0^\infty y^{s+\frac{\ell}{2}}\cdot\Theta_\ell(y)\,\frac{dy}{y} \;=\; \frac{1}{4}\sum_{0\neq\alpha}\alpha^\ell\cdot\int_0^\infty y^{s+\frac{\ell}{2}}\cdot e^{-\pi y|\alpha|^2}\,\frac{dy}{y} \;=\; \frac{1}{4}\sum_{0\neq\alpha}\alpha^\ell\cdot\int_0^\infty y^{s+\frac{\ell}{2}}\cdot e^{-\pi y|\alpha|^2}\,\frac{dy}{y}$$

$$=\; \frac{\pi^{-s}}{4}\sum_{0\neq\alpha}\frac{\alpha^\ell}{|\alpha|^{2s+\ell}}\cdot\int_0^\infty y^{s+\frac{\ell}{2}}\cdot e^{-y}\,\frac{dy}{y} \;=\; \pi^{-(s+\frac{\ell}{2})}\Gamma(s+\frac{\ell}{2})\cdot\frac{1}{4}\sum_{0\neq\alpha}\frac{\alpha^\ell}{|\alpha|^{2s+\ell}}$$

$$=\; \pi^{-(s+\frac{\ell}{2})}\Gamma(s+\frac{\ell}{2})\cdot\frac{1}{4}\sum_{0\neq\alpha}\frac{\alpha^\ell}{|\alpha|^\ell}\cdot\frac{1}{|\alpha|^{2s}} \;=\; \pi^{-(s+\frac{\ell}{2})}\Gamma(s+\frac{\ell}{2})\,L(s,\chi_\ell)$$

A parallel computation works for $\ell<0$. Then break the integral into two pieces, from 0 to 1 and then 1 to ∞. The integral from 1 to ∞ converges nicely for *all* $s\in\mathbb{C}$, so gives an entire function. The integral from 0 to 1 is converted to one from 1 to ∞ by the change of variables $y\to 1/y$ and using the functional equation of Θ_ℓ:

$$\int_0^1 y^{s+\frac{|\ell|}{2}}\,\Theta_\ell(y)\,\frac{dy}{y} \;=\; \int_1^\infty y^{-(s+\frac{|\ell|}{2})}\,\Theta_\ell(1/y)\,\frac{dy}{y} \;=\; \int_1^\infty y^{-(s+\frac{|\ell|}{2})}\,y^{|\ell|+1}\Theta_\ell(y)\,\frac{dy}{y}$$

$$=\; \int_1^\infty y^{1-s+\frac{|\ell|}{2})}\,\Theta_\ell(y)\,\frac{dy}{y}$$

Thus,

$$\pi^{-(s+\frac{|\ell|}{2})}\Gamma(s+\frac{|\ell|}{2})\,L(s,\chi_\ell) \;=\; \int_1^\infty (y^{s+\frac{|\ell|}{2}}+y^{1-s+\frac{|\ell|}{2})})\,\Theta_\ell(y)\,\frac{dy}{y}$$

The right-hand side converges very well, so is entire. ///

[10.10] With $\chi_\ell(\alpha)=(\alpha/|\alpha|)^\ell$, and the L-functions $L(s,\chi)$ as in the previous example, express $L(4,\chi_{-8})$ as a polynomial in $L(2,\chi_{-4})$.

The specific arguments make these L-function values be essentially the *Eisenstein series* attached to the lattice of Gaussian integers:

$$L(k,\chi_{-2k}) \;=\; \frac{1}{4}\sum_{0\neq\lambda\in\mathbb{Z}[i]}\frac{(\lambda/|\lambda|)^{-2k}}{|\lambda|^{2k}} \;=\; \frac{1}{4}\sum_{0\neq\lambda\in\mathbb{Z}[i]}\frac{1}{\lambda^{2k}} \;=\; \frac{1}{4}\cdot E_{2k}(\Lambda)$$

One charming way to compare $E_4(\Lambda)$ and $E_8(\Lambda)$ is by using the rigid behavior of *modular forms*, which proves that up to constants there is a unique holomorphic elliptic modular form of weight 8 (for $SL_2(\mathbb{Z})$). Certainly E_8 is such, and E_4^2 is, also. Thus, E_8 is a constant multiple of E_4^2. With lattice $\Lambda=\mathbb{Z}z+\mathbb{Z}$, in this normalization Eisenstein series have *Fourier expansions*

$$E_{2k}(z) \;=\; 2\zeta(2k)+\frac{(2\pi i)^{2k}}{(2k-1)!}\sum_{n\geq 1}\sigma_{2k-1}(n)\,e^{2\pi inz}$$

Thus, making the leading coefficient 1,

$$\left(\frac{E_4}{2\zeta(4)}\right)^2 \;=\; \frac{E_8}{2\zeta(8)}$$

so

$$E_8 \;=\; E_4^2 \cdot \frac{\zeta(8)}{\zeta(4)^2}$$

Note that the constant is in fact *rational*. ///

[10.11] Show how to achieve the effect of replacing a *quartic* by a *cubic* in an elliptic integral: exhibit a change of variables so that

$$\int_a^b \frac{dx}{\sqrt{x^4-1}} \;=\; \int_A^B \frac{dy}{\sqrt{4y^3+6y^2+4y+1}}$$

The trick is to use a linear fractional transformation to move one of the zeros of the quartic to ∞. For example, $g = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ sends 1 to ∞. Replacing x by $g \cdot y$ gives

$$\int \frac{d\big(\frac{y}{-y+1}\big)}{\sqrt{\big(\frac{y}{-y+1}\big)^4 - 1}} \;=\; \int \frac{\frac{1}{(-y+1)^2}\;dy}{\sqrt{\big(\frac{y}{-y+1}\big)^4 - 1}} \;=\; \int \frac{dy}{\sqrt{y^4 - (-y+1)^4}}$$

Replace y further by $y+1$, to obtain

$$\int \frac{dy}{\sqrt{(y+1)^4 - y^4}} \;=\; \int \frac{dy}{\sqrt{4y^3+6y^2+4y+1}}$$

as desired. ///

[10.12] Fix a lattice L. Express

$$f(z) \;=\; \frac{1}{z^4} + \sum_{0\neq\lambda\in L} \frac{1}{(z-\lambda)^4}$$

in terms of $\wp(z)$ and $\wp'(z)$.

This function is *even*, so we anticipate it is expressible in terms of $\wp(z)$. To obtain the expression, we try to cancel the poles, leaving an entire, doubly-periodic functions, which must be constant, by Liouville. The Laurent expansions of f and $\wp$ at 0 are of the forms

$$f(z) \;=\; \frac{1}{z^4} + a + bz^2 + \ldots \qquad\qquad \wp(z) \;=\; \frac{1}{z^2} + Bz^2 + \ldots$$

since the convergence trick for $\wp(z)$ makes the constant 0. In the proof of the Weierstraß equation, one discovers that B is essentially an Eisenstein series

$$E_{2k}(\Lambda) \;=\; \sum_{0\neq\lambda\in\Lambda} \frac{1}{\lambda^{2k}}$$

Precisely, $B = 3E_4$. Similarly,

$$a \;=\; \Big(f(z) - \frac{1}{z^4}\Big)\Big|_{z=0} \;=\; E_4$$

Thus,

$$\wp(z)^2 \;=\; \frac{1}{z^4} + 2B + O(z^2)$$

so

$$f(z) - \wp(z)^2 \;=\; (a - 2B) + O(z^2)$$

Thus,

$$f(z) - \wp(z)^2 \;=\; a - 2B + O(z^2)$$

and by Liouville

$$f(z) - \wp(z)^2 \;=\; a - 2B \;=\; E_4 - 6E_4 \;=\; -5E_4$$

[10.13] Express $\wp(2z)$ in terms of $\wp(z)$.

Let ω_1, ω_2 be a basis for the lattice Λ. Since $\wp(2z)$ has double poles at $\Lambda/2$, with residues $1/4$, $\wp(2z) - \frac{1}{4}\wp(z)$ has double poles exactly at $\frac{\omega_1}{2} + \Lambda$, $\frac{\omega_2}{2} + \Lambda$, and $\frac{\omega_1+\omega_2}{2} + \Lambda$.

Let a be any one of $\omega_1/2$, $\omega_2/2$, or $(\omega_1 + \omega_2)/2$. Since $\wp(z) - \wp(a)$ is still *even*, and has a zero at $z = a$, this is a *double* zero. Since the number of poles is equal to the number of zeros, and $\wp(z)$ has a double pole, there are no other zeros of $\wp(z) - \wp(a)$. Thus,

$$\big(\wp(2z) - \frac{1}{4}\,\wp(z)\big) \cdot \big((\wp(z) - \wp(\frac{\omega_1}{2}))(\wp(z) - \wp(\frac{\omega_2}{2}))(\wp(z) - \wp(\frac{\omega_1 + \omega_2}{2}))\big)$$

is an even function that has poles only on the lattice.

Again, the Laurent expansion of $\wp(z)$ is the power series of $\wp(z) - 1/z^2$ plus $1/z^2$, and the coefficients of the power series can be computed via derivatives, giving

$$\frac{1}{z^2} + 3E_4 z^2 + 5E_6 z^4 + 7E_8 z^6 + \ldots$$

Thus,

$$\wp(2z) - \frac{1}{4}\,\wp(z) \;=\; 3E_4(2^2 - 1)z^2 + 5E_6(2^3 - 1)z^4 + 7E_8(2^8 - 1)z^6 + \ldots$$

Thus, the (at least) double zero partly cancels the order-six pole, and the order of pole of the adjusted function is at most 4. Via Liouville's theorem, it is inevitably a *polynomial* in $\wp(z)$, of degree at most 2:

$$\big(\wp(2z) - \frac{1}{4}\,\wp(z)\big) \cdot \big((\wp(z) - \wp(\frac{\omega_1}{2}))(\wp(z) - \wp(\frac{\omega_2}{2}))(\wp(z) - \wp(\frac{\omega_1 + \omega_2}{2}))\big) \;=\; A\wp(z)^2 + B\wp(z) + C$$

This leaves 3 constants A, B, C to be determined. The leading $1/z^4$ coefficient of the Laurent expansion is the z^2 coefficient of $\wp(2z) - \wp(z)/4$, namely, $9E_4$. This is the constant A, so

$$\big(\wp(2z) - \frac{1}{4}\,\wp(z)\big) \cdot \big((\wp(z) - \wp(\frac{\omega_1}{2}))(\wp(z) - \wp(\frac{\omega_2}{2}))(\wp(z) - \wp(\frac{\omega_1 + \omega_2}{2}))\big) - 9E_4\wp(z)^2 \;=\; B\wp(z) + C$$

Evaluating the equation at a zero z_o of $\wp(z)$ gives

$$-\wp(2z_o) \cdot \wp(\frac{\omega_1}{2}) \cdot \wp(\frac{\omega_2}{2}) \cdot \wp(\frac{\omega_1 + \omega_2}{2}) \;=\; C$$

If $\wp(2z_o)$ has a pole at $z = z_o$, then z_o is among the 2-division-point values $\omega_1/2, \omega_2/2, (\omega_1 + \omega_2)/2$, and this evaluation requires a little more effort (which we do not exert here).

To determine B, we could take a derivative of the equation and evaluate anywhere $\wp'(z) \neq 0$, ...

If we cared more, we could pursue this or various other possibilities, such as multiplying out the Laurent expansions at 0 and comparing terms. ///

[10.14] Show that

$$\theta(z) \;=\; \sum_{v\in\mathbb{Z}^8} e^{\pi i|v|^2\cdot z} \qquad\qquad (\text{with } z\in\mathfrak{H})$$

is an elliptic modular form of weight 4 for the congruence subgroup Γ_θ.

We need to use the fact that the subgroup Γ_θ is generated by $z\to z+2$ and $z\to -1/z$. The invariance of $\theta(z)$ under $z\to z+2$ is clear, since each term is unchanged. Taking $z=iy$ with $y>0$, Poisson summation proves $\theta(iy)=\frac{1}{(iy)^4}\,\theta(i/y)$. By the identity principle, $\theta(-1/z)=z^4\cdot\theta(z)$, which is the correct behavior for a weight-four modular form. ///
