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Complex analysis examples 10

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[This document is http://www.math.umn.edu/~garrett/m/complex/examples_2014-15/cx_ex_10.pdf]

If you want feedback from me on your treatment of these examples, please get your work to me by Friday, Mar 27, preferably as a PDF emailed to me.

[10.1] Show that there is a well-defined, holomorphic function $1/\sqrt{1+z^4}$ on the region $|z| > 2$. Show that $\int_\gamma \frac{dz}{\sqrt{1+z^4}} = 0$, where γ traces out $|z| = 2$.

[10.2] Let γ be a simple closed path counter-clockwise encircling $0, 2$, and not enclosing -2. Let δ be a simple closed path counter-clockwise encircling $-2, 0$, and not enclosing 2. Show that there is a holomorphic function $1/\sqrt{z(z^2-4)}$ on the annulus $1 < |z-1| < 3$, and a holomorphic function $1/\sqrt{z(z^2-4)}$ on the annulus $1 < |z+1| < 3$. Show that the two *periods*

$$\int_\gamma \frac{dz}{\sqrt{z(z^2-4)}} \qquad\qquad \int_\delta \frac{dz}{\sqrt{z(z^2-4)}}$$

are *linearly independent* over $\mathbb{R}$.

[10.3] Show that for irrational $\alpha \in \mathbb{R}$, the set $\{m + n\alpha : m, n \in \mathbb{Z}\}$ is *dense* in $\mathbb{R}$.

[10.4] Let $v_1, \ldots, v_n$ be linearly independent vectors in $\mathbb{R}^n$, and $L = \mathbb{Z}v_1 + \ldots + \mathbb{Z}v_n$ the lattice generated by them. Let $\mathbb{R}^n$ have its usual inner product and associated metric. For $r > 0$ let B_r be the ball of radius 0 centered at $0 \in \mathbb{R}^n$. Show that for small-enough $r > 0$ we have $B_r \cap L = \{0\}$.

[10.5] Let L be a *lattice* in $\mathbb{R}^n$, that is, the $\mathbb{Z}$-module generated by n vectors linearly independent over $\mathbb{R}$. Prove that

$$\sum_{0 \neq \lambda \in L} \frac{1}{|\lambda|^s}$$

is absolutely convergent for $\mathrm{Re}(s) > n$, where $|\cdot|$ is the usual *length* in $\mathbb{R}^n$. (Do not invoke any non-existent *integral tests* in several variables, despite the fact that the idea of such gives a good heuristic.)

[10.6] Recall that we need *finite growth order* $|f(z)| \ll e^{|z|^N}$ as $|z| \to +\infty$ in a strip $a \le \mathrm{Re}(z) \le b$, for *some* N, before we can invoke the Phragmén-Lindelöf theorem. Use the integral representation of $\zeta(s)$ via $\theta(y)$, and properties of $\Gamma(s)$, to show that it has finite order of growth in $-1 \le \mathrm{Re}(s) \le 2$.

[10.7] Show that $f(x, y) = (x \pm iy)^\ell \, e^{-\pi(x^2+y^2)}$ is multiplied by $i^{-\ell}$ by Fourier transform

$$\widehat{f}(\xi, \eta) \;=\; \int_{\mathbb{R}^2} e^{-2\pi i(\xi x + \eta y)} \, f(x, y) \, dx, dy$$

Hint: rewrite this in terms of $z = x + iy$ and $\overline{z}$, and another complex variable $w = \xi + i\eta$ and $\overline{w}$, and look for a chance to differentiate under the integral defining the Fourier transform.

[10.8] Define a *harmonic theta function* $\Theta_\ell(y)$ by

$$\Theta_\ell(y) \;=\; \begin{cases} \dfrac{1}{4} \displaystyle\sum_{(0,0) \neq (m,n) \in \mathbb{Z}^2} (m + in)^\ell \, e^{-\pi y(m^2+n^2)} & (\text{for } \ell > 0) \\ \dfrac{1}{4} \displaystyle\sum_{(0,0) \neq (m,n) \in \mathbb{Z}^2} (m - in)^{|\ell|} \, e^{-\pi y(m^2+n^2)} & (\text{for } \ell < 0) \end{cases}$$

Show that this is identically 0 unless ℓ is divisible by 4, and prove the *functional equation*

$$\Theta_\ell(1/y) \;=\; y^{\ell+1} \cdot \Theta_\ell(y)$$

[10.9] Let $\chi(\alpha) = (\alpha/|\alpha|)^\ell$ for $\alpha \in \mathbb{C}^\times$. The associated *Hecke L-function* on the Gaussian integers $\mathbb{Z}[i]$ is

$$L(s,\chi) \;=\; \frac{1}{\#\mathbb{Z}[i]^\times} \sum_{0\neq\alpha\in\mathbb{Z}[i]} \frac{\chi(\alpha)}{|\alpha|^{2s}} \;=\; \frac{1}{4} \sum_{0\neq\alpha\in\mathbb{Z}[i]} \frac{\chi(\alpha)}{|\alpha|^{2s}}$$

Show that this is identically 0 unless ℓ is divisible by 4. Prove that $L(s,\chi_\ell)$ has an analytic continuation and functional equation and has the integral representation

$$\pi^{-(s+\frac{|\ell|}{2})}\,\Gamma\Big(s+\frac{|\ell|}{2}\Big)\,L(s,\chi) \;=\; \int_0^\infty y^{s+\frac{|\ell|}{2}}\,\Theta_\ell(y)\,\frac{dy}{y} \qquad \text{(for } \mathrm{Re}\,(s) > 1)$$

[10.10] With $\chi_\ell(\alpha) = (\alpha/|\alpha|)^\ell$, and the L-functions $L(s,\chi)$ as in the previous example, express $L(4,\chi_{-8})$ as a polynomial in $L(2,\chi_{-4})$.

[10.11] Show how to achieve the effect of replacing a *quartic* by a *cubic* in an elliptic integral: exhibit a change of variables so that

$$\int_a^b \frac{dx}{\sqrt{x^4-1}} \;=\; \int_A^B \frac{dy}{\sqrt{4y^3+6y^2+4y+1}}$$

[10.12] Fix a lattice L. Express

$$f(z) \;=\; \frac{1}{z^4} + \sum_{0\neq\lambda\in L} \frac{1}{(z-\lambda)^4}$$

in terms of $\wp(z)$ and $\wp'(z)$.

[10.13] Express $\wp(2z)$ in terms of $\wp(z)$.

[10.14] Show that

$$\theta(z) \;=\; \sum_{v\in\mathbb{Z}^8} e^{\pi i|v|^2\cdot z} \qquad \text{(with } z \in \mathfrak{H})$$

is an elliptic modular form of weight 4 for the congruence subgroup Γ_θ.
