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Complex analysis midterm discussion03

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/˜garrett/

[03.1] Give an explicit conformal map of the half-disk $\{z = x + iy : |z| < 1,\ x > 0\}$ to the unit disk $\{z : |z| < 1\}$.

These are both non-degenerate bi-gons, so we know this can be accomplished by a composite of linear fractional transformations and power maps $z \to z^\alpha$.

First, map one of the vertices $\pm i$ to ∞, and the other to 0, by $z \to \frac{z+i}{z-i}$, for example. To determine the images of the sides, it suffices to track a third point on each, in addition to $\pm i$. For the vertical straight line segment from $-i$ to $+i$ use the third point 0, which maps to -1. Thus, that segment maps to the ray along the negative real axis. For the half-circle, use third point 1, which maps to $(1+i)/(1-i) = i$, so this side maps to the positive imaginary axis.

Rotate clockwise by $\pi/2$ radians, by multiplying by $-i$, to put one edge on the *positive* real axis, so that the bi-gon becomes the interior of the first quadrant. Use $z \to z^2$ to map the first quadrant to the upper half-plane, and then the inverse Cayley map $z \to \frac{z-i}{-iz+1}$ to map to the disk.

Altogether, this is

$$z \longrightarrow \frac{z+i}{z-i} \longrightarrow -i\frac{z+i}{z-i} \longrightarrow \Big(-i\frac{z+i}{z-i}\Big)^2 \longrightarrow \frac{\Big(-i\frac{z+i}{z-i}\Big)^2 - i}{-i\Big(-i\frac{z+i}{z-i}\Big)^2 + 1}$$

$$= \frac{-(z+i)^2 - i(z-i)^2}{i(z+i)^2 + (z-i)^2} = \frac{-z^2 - 2iz + 1 - iz^2 - 2z + i}{iz^2 - 2z - i + z^2 - 2iz - 1} = \frac{-(1+i)z^2 - 2(1+i)z + (1+i)}{(1+i)z^2 - 2(1+i)z - (1+i)}$$

$$= \frac{-z^2 - 2z + 1}{z^2 - 2z - 1}$$

mapping the half-disk to the disk. ///

[03.2] Determine a finite set $S \subset \mathbb{C}$ of points such that for $w_o \notin S$ there is a holomorphic function $f(w)$ near w_o such that $z = f(w)$ gives a solution to the equation $z^5 - 5z - w = 0$. (*Hint:* holomorphic inverse function theorem.)

In a relation $F(z) = w$ with holomorphic F, the holomorphic inverse function theorem can only fail at points z_o where $F'(z_o) = 0$. In the case at hand, $F(z) = z^5 - 5z$, $F'(z) = 5(z^4 - 1)$, so the inverse function theorem can only fail at $z_o = \pm 1, \pm i$. The corresponding values of $w = F(z_o)$ are

$$w_o = z_0^5 - 5z_o = z_o(z_o^4 - 1) + 4z_o = 4z_o = \pm 4,\ \pm 4i \qquad (\text{for } z_o^4 = 1)$$

Thus, excluding $S = \{\pm 4,\ \pm 4i\}$ ensures a local holomorphic inverse. ///

[03.3] Show that $f(z) = e^{iz} - z$ has at least one complex zero.

One approach is by the *argument principle*: the net change of the *argument* of f around a large-enough box, with vertices $\pm T \pm iT$, is 2π times the number of zeros inside (assuming that T is adjusted so that there are no zeros exactly on the rectangle: this adjustment is possible, by the identity principle).

Along the top side, $|e^{i(x+iT)}| = e^{-T}$, which is (much!) smaller than $|x+iT|$ for $T \geq 1$, for example. Thus, along that top edge, the argument of $e^{iz} - z$ stays within $\pi/2$ of that of $z = x+iT$. Thus, while $\arg z$ goes from $\pi/4$ to $3\pi/4$ as $z = x+iT$ goes from $T+iT$ to $-T+iT$, the argument of $e^{iz} - z$ can *at most* have net change

$$\Big(\frac{3\pi}{4} + \frac{\pi}{2}\Big) - \Big(\frac{\pi}{4} - \frac{\pi}{2}\Big) \;=\; \frac{3\pi}{2}$$

and *at least* by

$$\Big(\frac{3\pi}{4} - \frac{\pi}{2}\Big) - \Big(\frac{\pi}{4} + \frac{\pi}{2}\Big) \;=\; -\frac{\pi}{2}$$

In any case, it is $O(1)$, in Landau's notation.

Along the bottom side, $|e^{i(x+iT)}| = e^{T}$, which is (much!) larger than $|\pm T \pm iT|$ for $T \geq 6$, for example. Thus, along the bottom edge, the argument of $e^{iz} - z$ stays within $\pi/2$ of that of e^{iz}. Thus, while $\arg e^{i(x-iT)} = x$ goes from $-T$ to $+T$, the argument of $e^{iz} - z$ changes at most by

$$\Big(T + \frac{\pi}{2}\Big) - \Big(-T - \frac{\pi}{2}\Big) \;=\; 2T + O(1)$$

and *at least* by

$$\Big(T - \frac{\pi}{2}\Big) - \Big(-T + \frac{\pi}{2}\Big) \;=\; 2T + O(1)$$

Along the vertical sides, use the trick that the absolute value of the net change in argument is at most $2\pi(q+1)$ where q is the number of times the *real part* vanishes. Further, slightly adjust T so that $\cos T = 0$, so that

$$\mathrm{Re}\big(e^{i(T\pm iy)} - (T \pm iy)\big) \;=\; \cos T - T \;=\; -T$$

That is, the real part does not vanish at all along the vertical edges, so the net changes are bounded by $\pm 2\pi = O(1)$.

Putting these together, for large-enough T, the net change in the argument around the $\pm T \pm iT$ rectangle is $2T + O(1)$, so the number of zeros inside is $\frac{T}{\pi} + O(1)$. For large-enough T, this is greater than 1, so there is at least one zero inside. ///
