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Complex analysis midterm discussion 04

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[This document is
http://www.math.umn.edu/˜garrett/m/complex/examples_2014-15/midterm_discussion_04.pdf]

[04.1] Give a harmonic function u on the annulus $\frac{1}{2} \leq |z| \leq 2$ such that on the *outer* boundary circle $|z| = 2$ the boundary-value function is $u(2e^{i\theta}) = e^{i\theta}$, while on the *inner* boundary circle $|z| = \frac{1}{2}$ the boundary-value function is $u(\frac{1}{2}e^{i\theta}) = e^{-i\theta}$.

Recall that we showed that all harmonic functions on an annulus are of the form

$$a_o + b_o \log |z| + \sum_{0 \neq n \in \mathbb{Z}} a_n z^n + b_n \overline{z}^b$$

At the very least, one should remember that $\log |z|$ is harmonic, and all the other terms are either holomorphic or anti-holomorphic, so certainly harmonic.

We might also break the problem into two pieces, namely, making one harmonic function u_1 that is $e^{i\theta}0$ on the outer circle while 0 on the inner, and making a second harmonic function u_2 that is 0 on the outer circle and $e^{-i\theta}$ on the inner.

For the On circles $z = re^{i\theta}$, both z^n and $\overline{z}^{-n}$ give constant multiples of $e^{in\theta}$, for any $n \in \mathbb{Z}$. Thus, any $az + b\overline{z}^{-1}$ will give a multiple of $e^{i\theta}$ on every circle. To make the multiple be exactly $e^{i\theta}$ on $|z| = 2$ and 0 on $|z| = \frac{1}{2}$ is to require

$$\begin{cases} a \cdot 2 + b \cdot \frac{1}{2} & = & 1 \\ a \cdot \frac{1}{2} + b \cdot 2 & = & 0 \end{cases}$$

which gives $a = 8/15$, $b = -2/15$, so

$$u_1 \;=\; \frac{8}{15} z - \frac{2}{15} \overline{z}^{-1}$$

Similarly, for u_2, every expression $az^{-1} + b\overline{z}$ will be a constant multiple of $e^{-i\theta}$ on circles. To make the multiple be exactly 0 on $|z| = 2$ and $e^{-i\theta}$ on $|z| = Hf$ is to require

$$\begin{cases} a \cdot \frac{1}{2} + b \cdot 2 & = & 0 \\ a \cdot 2 + b \cdot \frac{1}{2} & = & 1 \end{cases}$$

which is the same system as for the constants for u_1. Thus,

$$u_2 \;=\; \frac{8}{15} z^{-1} - \frac{2}{15} \overline{z}$$

Putting the two together, the desired harmonic function on the annulus is

$$\frac{8}{15} z - \frac{2}{15} \overline{z}^{-1} + \frac{8}{15} z^{-1} - \frac{2}{15} \overline{z}$$

[04.2] Show that for $t \in \mathbb{R}$

$$\left|\Gamma(\tfrac{1}{2} + it)\right|^2 \;=\; \frac{2\pi}{e^{\pi t} + e^{-\pi t}}$$

It is reasonable to imagine that this would follow from the relation

$$\Gamma(s) \cdot \Gamma(1-s) \;=\; \frac{\pi}{\sin \pi s}$$

and then we can contemplate how much of the proof of this relation we might want to recall.

Taking $s = \frac{1}{2} + it$ with real t, recalling that $\overline{\Gamma(s)} = \Gamma(\overline{s})$,

$$|\Gamma(\tfrac{1}{2}+it)|^2 \;=\; \Gamma(\tfrac{1}{2}+it)\cdot\overline{\Gamma(\tfrac{1}{2}+it)} \;=\; \Gamma(\tfrac{1}{2}+it)\cdot\Gamma(\overline{\tfrac{1}{2}+it}) \;=\; \Gamma(\tfrac{1}{2}+it)\cdot\Gamma(\tfrac{1}{2}-it)$$

$$=\; \Gamma(\tfrac{1}{2}+it)\cdot\Gamma(1-(\tfrac{1}{2}+it)) \;=\; \frac{\pi}{\sin\pi(\frac{1}{2}+it)}$$

Then

$$\sin\pi(\tfrac{1}{2}+it) \;=\; \frac{e^{i\pi(\frac{1}{2}+it)} - e^{-i\pi(\frac{1}{2}+it)}}{2i} \;=\; \frac{e^{i\pi/2}\cdot e^{-\pi t} - e^{-i\pi/2}\cdot e^{\pi t}}{2i} \;=\; \frac{e^{-\pi t}+e^{\pi t}}{2}$$

So

$$|\Gamma(\tfrac{1}{2}+it)|^2 \;=\; \frac{\pi}{\sin\pi(\frac{1}{2}+it)} \;=\; \frac{2\pi}{e^{-\pi t}+e^{\pi t}}$$

That *reflection identity* can be proven in at least two ways. One way is to use Euler's integral for both $\Gamma(s)$ and $\Gamma(1-s)$ in the range $0 < \mathrm{Re}(s) < 1$, change variables, and then use the Hankel contour, as in the notes. Another way, feasible after we have the Hadamard product facts in hand, if we somehow believe that $\Gamma(s)$ has no 0s, and is of growth-order $\lambda = 1$, is to use the product expansion

$$\frac{1}{\Gamma(s)} \;=\; e^{a+bs}\prod_{n=1}^{\infty}\Big(1+\frac{s}{n}\Big)e^{-s/n}$$

and use $\Gamma(1-s) = -s\Gamma(-s)$:

$$\frac{1}{\Gamma(s)\,\Gamma(1-s)} \;=\; \frac{-s}{\Gamma(s)\,\Gamma(-s)} \;=\; -se^{a+bs}\prod_{n=1}^{\infty}\Big(1+\frac{s}{n}\Big)e^{-s/n}\cdot e^{a-bs}\prod_{n=1}^{\infty}\Big(1+\frac{-s}{n}\Big)e^{s/n}$$

$$=\; -e^{2a}\cdot s\prod_{n=1}^{\infty}\Big(1-\frac{s^2}{n^2}\Big) \;=\; \text{constant}\cdot\sin\pi s$$

by grouping terms. To determine the constant, take $s = \frac{1}{2}$, and recall the usual trick:

$$\Gamma(\tfrac{1}{2})^2 \;=\; \Big(\int_0^\infty e^{-t}\,t^{\frac{1}{2}}\,\frac{dt}{t}\Big)^2 \;=\; \Big(\int_0^\infty e^{-t^2}\,t\,\frac{2t\,dt}{t^2}\Big)^2 \;=\; \Big(\int_{-\infty}^\infty e^{-t^2}\,dt\Big)^2$$

$$=\; \int_{\mathbb{R}^2} e^{-(x^2+y^2)}\,dx\,dy \;=\; \int_0^\infty\int_0^{2\pi} e^{-r^2}\,d\theta\,r\,dr \;=\; 2\pi\int_0^\infty e^{-r^2}\,r\,dr \;=\; \pi\int_0^\infty e^{-r^2}\,2r\,dr$$

$$=\; \pi\int_0^\infty e^{-u}\,du \;=\; \pi$$

Thus,

$$\frac{1}{\Gamma(s)\,\Gamma(1-s)} \;=\; \frac{\sin\pi s}{\pi}$$

[04.3] Prove that

$$\prod_{n\ge 3}\Big(1+\frac{1}{n\log n}\Big) \;=\; +\infty \qquad\text{and}\qquad \prod_{n\ge 3}\Big(1-\frac{1}{n\log n}\Big) \;=\; 0$$

For the first, let's prove by induction that

$$\prod_{n=1}^{N}(1+a_n) \;\geq\; 1+a_1+a_2+\ldots+a_N \qquad\qquad (\text{for } a_n>0)$$

The assertion is $1+a_1 \geq 1+a_1$ for $N=1$. By induction,

$$\prod_{n=1}^{N+1}(1+a_n) \;=\; (1+a_{N+1})\cdot\prod_{n=1}^{N}(1+a_n) \;\geq\; (1+a_{N+1})\cdot\Big(1+a_1+\ldots+a_N\Big)$$

$$=\; 1+a_1+\ldots+a_{N+1}+a_{N+1}\Big(a_1+\ldots+a_N\Big) \;\geq\; 1+a_1+\ldots+a_{N+1}$$

by the positivity assumption. Thus, using the monotonicity of $1/t\log t$,

$$\prod_{3\leq n\leq N}\Big(1+\frac{1}{n\log n}\Big) \;\geq\; 1+\sum_{3\leq n\leq N}\frac{1}{n\log n} \;\geq\; \int_3^N \frac{dt}{t\log t} \;=\; \log\log N-\log\log 3 \;\geq\; \log\log N$$

This goes to $+\infty$ as $N\to+\infty$, so the infinite product has value $+\infty$.

For the second product, use

$$\log(1-a) \;=\; -\Big(a+\frac{a^2}{2}+\frac{a^3}{3}+\ldots\Big) \;\leq\; -a \qquad\qquad (\text{for } 0<a<1)$$

so

$$\log\prod_{3\leq n\leq N}\Big(1-\frac{1}{n\log n}\Big) \;=\; \sum_{3\leq n\leq N}\log\Big(1-\frac{1}{n\log n}\Big) \;\leq\; -\sum_{3\leq n\leq N}\frac{1}{n\log n}$$

$$\leq\; -\int_3^N\frac{dt}{t\log t} \;\leq\; -(\log\log N-\log\log 3)$$

again by monotonicity of $1/t\log t$. This goes to $-\infty$ as $N\to+\infty$, so the product itself goes to 0. ///
