(*September 17, 2020*)

Complex analysis examples discussion 01

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[This document is
http://www.math.umn.edu/~garrett/m/complex/examples_2020-21/cx_discussion_01.pdf]

[01.1] Express the two values for $\sqrt{i}$ in terms of radicals.

The definition of square root here is that $(a+bi)^2 = i$. Multiplying out gives $(a^2-b^2)+2abi = i$. Separating into real and imaginary parts gives $a^2-b^2 = 0$ and $2ab = 1$. From the first of these, $b = \pm a$, and substituting into the second gives $\pm 2a^2 = 1$. Thus,

$$\pm\sqrt{i} \;=\; \pm\Big(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$$

Another approach, making more use of the geometry via the Euler identity $e^{ix} = \cos x + i\sin x$, is as follows, although a careful version of this is more verbose in this context. First, using that Euler identity, $e^ix = i$ exactly for $x = \frac{\pi}{2} + 2\pi n$ for integer n. Then $(e^{iy})^2 = i$ implies $e^{2iy} = e^{i(\frac{\pi}{2}+2\pi n)}$ exactly for integers n. That is, again by Euler's identity,

$$2iy \;=\; \frac{\pi}{2} - 2\pi n \qquad\qquad \text{(for integers } n)$$

and

$$\pm\sqrt{i} \;=\; e^{i(\frac{\pi}{4}+\pi n)} \qquad\qquad \text{(for integers } n)$$

Since $e^{\pi i n} = 1$ exactly for integers n, the complete list of *distinct* values is

$$\pm\sqrt{i} \;=\; e^{i(\frac{\pi}{4})},\; e^{i(\frac{\pi}{4}+\pi)} \;=\; \cos\frac{\pi}{4} + i\sin\frac{\pi}{4},\; -\Big(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\Big) \;=\; \pm\Big(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\Big)$$

[01.2] Determine all values of i^i.

For this example, there is no alternative but to use Euler's identity, to raise complex numbers to the i^{th} power. That is, there is no alternative algebraic or limiting definition of α^i, unlike the case of integer, rational, or real exponents. From Euler's identity,

$$i \;=\; e^{\frac{\pi i}{2}+2\pi i n} \qquad\qquad \text{(for every integer } n)$$

Then

$$i^i \;=\; e^{(\frac{\pi i}{2}+2\pi i n)\cdot i} \;=\; e^{-\frac{\pi i}{2}-2\pi n} \qquad\qquad \text{(for every integer } n)$$

It is striking that the outcome is a collection of real numbers approaching 0^+ at one end and going to $+\infty$ at the other.

[01.3] Derive the usual formula for $\sin(z+w)$ by using e^z.

Use $\sin z = \frac{e^{iz}-e^{-iz}}{2i}$ and $\cos z = \frac{e^{iz}+e^{-iz}}{2}$, which hold for complex z, as well, and also the opposite formula $e^{iz} = \cos z + i\sin z$. Also note that $\cos z$ is *even*, while $\sin z$ is *odd*. Then the basic property $e^{z+w} = e^z \cdot e^w$ and elementary algebra give the desired identity:

$$\sin(z+w) \;=\; \frac{e^{i(z+w)} - e^{-i(z+w)}}{2i} \;=\; \frac{e^{iz}\,e^{iw} - e^{-iz}\,e^{-iw)}}{2i}$$

$$= \frac{(\cos z + i\sin z)\,(\cos w + i\sin w) - (\cos z - i\sin z)\,(\cos w - i\sin w)}{2i}$$

$$= \frac{2i\sin z\,\cos w + 2i\cos z\,\sin w}{2i} \;=\; \sin z\,\cos w + \cos z\,\sin w$$

[01.4] Express $\cos 5x$ as a polynomial in $\cos x$ and $\sin x$.

Again use $\sin z = \frac{e^{iz}-e^{-iz}}{2i}$ and $\cos z = \frac{e^{iz}+e^{-iz}}{2}$, $e^{iz} = \cos z + i\sin z$, and the even-ness of $\cos x$ and odd-ness of $\sin x$:

$$\cos 5x \;=\; \frac{e^{5ix}+e^{-5ix}}{2} \;=\; \frac{(e^{ix})^5+(e^{-ix})^5}{2} \;=\; \frac{(cosx + i\sin x)^5 + (\cos x - i\sin x)^5}{2}$$

$$= \frac{\sum_{j=0}^5 \binom{5}{j} \cos^j x\,(i\sin x)^{5-j} + \sum_{j=0}^5 \binom{5}{j} \cos^j x\,(-i\sin x)^{5-j}}{2}$$

$$= \sum_{\text{even } j=0}^{5} \binom{5}{j} \cos^j x\,(i\sin x)^{5-j} \;=\; \cos^4 x + 10\cos^2 x\,\sin^3 x + 4\cos x\,\sin^4 x$$

[01.5] By *mere algebra*, write a power series expansion near $z = 0$ for

$$f(z) \;=\; \frac{1}{(z-1)(z-2)}$$

This has to allow use of the geometric series expansion $\frac{1}{1-w} = 1 + w + w^2 + w^3 + \ldots$ for $|w| < 1$. It is convenient to separate the two factors of the denominator by a partial fractions expansion, as

$$\frac{1}{(z-1)(z-2)} \;=\; \frac{-1}{z-1} + \frac{1}{z-2}$$

The coefficients can be found by looking at the asymptotics as $z \to 1$ and $z \to 2$, for example. Thus,

$$\frac{1}{(z-1)(z-2)} \;=\; \frac{-1}{z-1} + \frac{1}{z-2} \;=\; \frac{1}{1-z} + \frac{-\frac{1}{2}}{1-\frac{z}{2}} \;=\; (1 + z + z^2 + \ldots) - \tfrac{1}{2}(1 + \frac{z}{2} + (\frac{z}{2})^2 + \ldots)$$

$$= \sum_{n=0}^{\infty} \Big(1 - \frac{1}{2^n}\Big)\, z^n$$

[01.6] Determine the radius of convergence of $\sum_{n\ge 1} \frac{3^n}{n(n+1)(n+2)}\, z^n$.

The ratio test succeeds:

$$\frac{3^n/n(n+1)(n+2)}{3^{n+1}/(n+1)(n+2)(n+3)} \;=\; \frac{1/n}{3/(n+3)} \;=\; \frac{n+3}{3n} \;=\; \frac{1}{3} + \frac{1}{n} \;\longrightarrow\; \frac{1}{3} \qquad (\text{as } n \to +\infty)$$

so the radius of convergence is $1/3$.

[01.7] Determine the radius of convergence of $\sum_{n\ge 1} \frac{n!}{n^n}\, z^n$.

The ratio test succeeds:

$$\frac{n!/n^n}{(n+1)!/(n+1)^{n+1}} \;=\; \frac{1/n^n}{(n+1)/(n+1)^{n+1}} \;=\; \Big(1 + \frac{1}{n}\Big)^n \;\longrightarrow\; e \qquad (\text{as } n \to +\infty)$$

by one characterization of the number e. That is, the radius of convergence is e.

[01.8] For two complex numbers a, b, with b not a non-positive integer, show that the radius of convergence of

$$\sum_{n\geq 0} \frac{a(a+1)(a+2)\ldots(a+n-1)(a+n)}{b(b+1)(b+2)\ldots(b+n-1)(b+n)}\, z^n$$

is at least 1.

If a happens to be a negative integer, the sum terminates, and is a polynomial, so has radius of convergence $+\infty$.

For a *not* a negative integer, the ratio test succeeds:

$$\frac{a(a+1)(a+2)\ldots(a+n-1)(a+n)}{b(b+1)(b+2)\ldots(b+n-1)(b+n)} \Big/ \frac{a(a+1)(a+2)\ldots(a+n)(a+n+1)}{b(b+1)(b+2)\ldots(b+n)(b+n+1)}$$

$$= \frac{b+n+1}{a+n+1} = \frac{1+\frac{b+1}{n}}{1+\frac{a+1}{n}} \longrightarrow 1 \qquad\qquad (\text{as } n\to+\infty)$$

so the radius of convergence is 1 in this case.

[01.9] From the very definition of convergence, show that when the partial sums of a series $a_1 + a_2 + \ldots$ are *bounded*, and when the elements of the sequence $\{b_n\}$ *are positive* (real) and *go to* 0 *monotonically*, then the series $\sum a_n b_n$ *converges*.

In fact, we show that the partial sums of $\sum_n a_n b_n$ form a *Cauchy* sequence, so, by the *completeness* of $\mathbb{C}$, the infinite sum converges. That is, we prove that the *tails* $\sum_{\ell=m}^n a_\ell b_\ell$ have the property that, given $\varepsilon > 0$, there is N such that $|\sum_{\ell=m}^n a_\ell b_\ell| < \varepsilon$ for all $m, n \geq N$.

To this end, we rearrange the tails by *partial summation*: letting $A_n = a_1 + \ldots + a_n$ and $B_n = b_1 + \ldots + b_n$,

$$a_m b_m + a_{m+1}b_{m+1} + \ldots + a_n b_n \;=\; (A_{m+1} - A_m)b_m + \ldots + (A_{n+1} - A_n)b_n$$

$$=\; -A_m b_m + A_{m+1}(b_m - b_{m+1}) + A_{m+2}(b_{m+1} - b_{m+2}) + \ldots + A_n(b_{n-1} - b_n) + A_{n+1}b_n$$

Taking absolute values, letting C be a bound for the $|A_n|$, using $b_j > b_{j+1} > 0$,

$$\Big| - A_m b_m + A_{m+1}(b_m - b_{m+1}) + \ldots + A_n(b_{n-1} - b_n) + A_{n+1}b_n \Big|$$

$$\leq\; Cb_m + C(b_m - b_{m+1}) + C(b_{m+1} - b_{m+2}) + \ldots + C(b_{n-1} - b_n) + Cb_n \;=\; Cb_m \;\longrightarrow\; 0$$

since $b_m \to 0$. Thus, the partial sums are Cauchy, and the infinite sum converges.

[01.10] Show that the function $f(z) = \sum z^n/n^2$ on the open disk $|z| < 1$ extends to a *continuous* function on the *closed* unit disc.

Since the power series has radius of convergence 1, it gives an infinitely-differentiable, hence continuous, function on the *open* disk. There are several possible arguments to prove continuity on the *closed* disk, but in any case it is important to note that Abel's theorem about *non-tangential* approach does not instantly prove this, since every neighborhood of a point on the boundary circle contains points *outside* the non-tangential approach regions address.

A general argument that applies here is that *a uniform limit of continuous functions is continuous.* The finite subsums $\sum_{n\leq N} z^n/n^2$ are *polynomials*, so certainly continuous. The differences are easily bounded, with $M \leq N$, with $|z| \leq 1$, by

$$\Big| \sum_{1\leq n\leq M} \frac{z^n}{n^2} - \sum_{1\leq n\leq N} \frac{z^n}{n^2} \Big| \;\leq\; \sum_{M<n\leq N} \frac{1}{n^2}$$

Since $\sum_{n\geq 1} 1/n^2$ converges, given $\varepsilon > 0$ there is M_o such that $\sum_{M<n\leq N} \frac{1}{n^2} < \varepsilon$ for all $M_o \leq M \leq N$. Thus, the partial sums of $\sum_n z^n/n^2$ for $|z| \leq 1$ form a Cauchy sequence in sup norm, so converge to a continuous function on the closed disk.
