(*October 16, 2020*)

Complex analysis examples discussion 02

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/˜garrett/

[This document is http://www.math.umn.edu/˜garrett/m/complex/examples_2020-21/cx_discussion_02.pdf]

[02.1] Parametrize counter-clockwise a circle γ of radius $r > 0$ centered at z_o, and *directly* compute $\int_\gamma (z-z_o)^n\,dz$ for all positive and negative integers n.

Discussion: Such a path can be parametrized as $\gamma(t) = z_o + re^{it}$ for $0 \le t \le 2\pi$. Then

$$\int_\gamma (z-z_o)^n\,dz \;=\; \int_0^{2\pi} (re^{it})^n\,d(re^{it}) \;=\; \int_0^{2\pi} (re^{it})^n\,ire^{it}\,dt$$

$$=\; ir^{n+1}\int_0^{2\pi} e^{i(n+1)t}\,dt \;=\; \begin{cases} \Big[it\Big]_0^{2\pi} & = \;\; 2\pi i \quad (\text{for } n=-1) \\ \Big[\dfrac{ir^{n+1}\cdot e^{i(n+1)t}}{i(n+1)}\Big]_0^{2\pi} & = \;\; 0 \quad (\text{for } n \ne -1) \end{cases}$$

as required. ///

[02.2] Of what rational function is $\sum_{n=0}^\infty n(n-1)(n-2)(n-3)z^n$ the power series expansion at 0?

Discussion: If the power of z were z^{n-4}, it might be even more obvious that the power series has something to do with the fourth derivative of $\sum z^n = \frac{1}{1-z}$. Yes, by an easy part of a theorem of Abel (and others), we can differentiate termwise to correctly find derivatives of power series within their radius of convergence. So

$$\sum n(n-1)(n-2)(n-3)z^n \;=\; z^4\cdot(\frac{d}{dz})^4 \sum z^n \;=\; z^4\cdot(\frac{d}{dz})^4\frac{1}{1-z} \;=\; z^4\cdot\frac{4!}{(1-z)^5}$$

as expected. ///

[02.3] Determine the Laurent expansions of $\dfrac{1}{1-z}$ in $|z|<1$, and in $|z|>1$. Observe that these two have no common region of convergence.

Discussion: When $|z|<1$, we have the iconic geometric series

$$\frac{1}{1-z} \;=\; \sum_{n=0}^\infty z^n$$

When $|z|>1$,

$$\frac{1}{1-z} \;=\; \frac{1}{z}\cdot\frac{1}{\frac{1}{z}-1} \;=\; -\frac{1}{z}\cdot\frac{1}{1-\frac{1}{z}} \;=\; -\frac{1}{z}\cdot\sum_{n=0}^\infty(\frac{1}{z})^n \;=\; -\sum_{n=0}^\infty(\frac{1}{z})^{n+1}$$

as required. ///

[02.4] Using only geometric series expansions, determine the Laurent expansion of $f(z) = 1/(z-1)(z-2)$ in the annulus $1<|z|<2$, and also in the annulus $|z|>2$.

Discussion: By partial fractions, for $1<|z|<2$, expanding geometric series,

$$\frac{1}{(z-1)(z-2)} \;=\; \frac{1}{z-2}-\frac{1}{z-1} \;=\; \frac{-\frac{1}{2}}{1-\frac{z}{2}} - \frac{1}{z}\cdot\frac{1}{1-\frac{1}{z}} \;=\; -\tfrac{1}{2}\Big(1+\frac{z}{2}+(\frac{z}{2})^2+\ldots\Big)-\frac{1}{z}\Big(1+\frac{1}{z}+\Big(\frac{1}{z}\Big)^2+\ldots\Big)$$

$$= -\tfrac{1}{2} - \sum_{n=1}^{\infty}(1 + \frac{1}{2^{n+1}})\, z^{-n} \qquad \text{(in the annulus } 1 < |z| < 2)$$

For $|z| > 2$, the $1/(z-2)$ requires slightly different treatment:

$$\frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1} = \frac{1}{z}\cdot\frac{1}{1-\frac{2}{z}} + \frac{1}{z}\cdot\frac{1}{1-\frac{1}{z}} = \frac{1}{z}\Big(1 + \frac{2}{z} + (\frac{2}{z})^2 + \ldots\Big) - \frac{1}{z}\Big(1 + \frac{1}{z} + \Big(\frac{1}{z}\Big)^2 + \ldots\Big)$$

$$= \sum_{n=1}^{\infty}(2^{n-1} - 1)\, z^{-n} = \sum_{n=2}^{\infty}(2^{n-1} - 1)\, z^{-n} \qquad \text{(in the annulus } 1 < |z| < 2)$$

as required. ///

[02.5] Determine the Laurent expansion of $f(z) = 1/(z-1)^3$ in the annulus $|z| > 1$, and in the annulus $|z-1| > 0$.

Discussion: In $|z| > 1$,

$$\frac{1}{z-1} = \frac{1}{z}\cdot\frac{1}{1-\frac{1}{z}} = \frac{1}{z}\cdot\Big(1 + \frac{1}{z} + \Big(\frac{1}{z}\Big)^2 + \ldots\Big) = \frac{1}{z} + \Big(\frac{1}{z}\Big)^2 + \ldots$$

Differentiating termwise two times gives

$$\frac{(-1)(-2)}{(z-1)^3} = \frac{(-1)(-2)}{z^3} + \frac{(-2)(-3)}{z^4} + \ldots + \frac{(-n)(-n-1)}{z^{n+2}} + \ldots$$

which simplifies to

$$\frac{1}{(z-1)^3} = \frac{1}{z^3} + \frac{2\cdot 3/5}{z^4} + \ldots + \frac{n(n+1)/2}{z^{n+2}} + \ldots$$

In the annulus $|z-1| > 0$, the given expression $f(z) = (z-1)^{-3}$ is already the Laurent expansion. ///

[02.6] Show that an entire function f satisfying $|f(z)| \le C\cdot(1+|z|)^{1/2}$ for some constant C is *constant*.

Discussion: This is a variant of Liouville's theorem and its proof. Since f is *entire*, its power series at 0

$$f(z) = \sum_{n\ge 0}\frac{f^{(n)}(0)}{n!}\cdot z^n$$

converges for all z and is equal to $f(z)$. For $0 \le n \in \mathbb{Z}$, use Cauchy's formula for derivatives, integrating counter-clockwise over a circle γ_R, of radius R, centered at 0:

$$f^{(n)}(0) = \frac{(n-1)!}{2\pi i}\int_{\gamma_R}\frac{f(w)\,dw}{(w-z)^{n+1}}$$

The easy/trivial estimate on the absolute value of the integral is that it is bounded by

$$\text{length } \gamma_R \times \text{max of integrand on } \gamma_R \le 2\pi R \times \frac{C\cdot(1+|R|)^{\frac{1}{2}}}{R^{n+1}} \le 2\pi R \times \frac{C\cdot(2R)^{\frac{1}{2}}}{R^{n+1}} \qquad \text{(for } R \ge 1)$$

which is dominated by $R^{\frac{3}{2}-(n+1)}$. For $n \ge 1$, this goes to 0 as $R \to +\infty$. Thus, all the power series coefficients but the 0^{th} are 0, and f is a constant. ///

[02.7] Show that an entire function f satisfying $|f(z)| \le C\cdot(1+|z|)^r$ for some $0 \le r \in \mathbb{R}$, and for some constant C, is a polynomial of degree at most r. (Yes, degrees of not-identically-zero polynomials are non-negative integers.)

Discussion: Another variant of Liouville's theorem. The argument basically recopies the previous, with a small difference at the end. Since f is *entire*, its power series at 0

$$f(z) \;=\; \sum_{n\geq 0} \frac{f^{(n)}(0)}{n!}\cdot z^n$$

converges for all z and is equal to $f(z)$. Use Cauchy's formula for derivatives, integrating counter-clockwise over a circle γ_R, of radius R, centered at 0:

$$f^{(n)}(0) \;=\; \frac{(n-1)!}{2\pi i}\int_{\gamma_R} \frac{f(w)\,dw}{(w-z)^{n+1}}$$

The easy/trivial estimate on the absolute value of the integral is that it is bounded by

$$\text{length } \gamma_R \times \text{max of integrand on } \gamma_R \;\leq\; 2\pi R \times \frac{C\cdot(1+|R|)^r}{R^{n+1}} \;\leq\; 2\pi R\times\frac{C\cdot(2R)^r}{R^{n+1}} \qquad (\text{for } R\geq 1)$$

which is dominated by $R^{(r+1)-(n+1)}$. For $n>r$, this goes to 0 as $R\to+\infty$. Thus, all the coefficients of z^n for $n>r$ are 0, and f is a polynomial of degree at most r. ///

[02.8] Show that an entire function f satisfying $|f(z)|\leq C\cdot\log(1+|z|)$ for some constant C is *constant.*

Discussion: Yet another variant of Liouville's theorem... ///

[02.9] Compute $\displaystyle\int_{-\infty}^{\infty}\frac{dx}{x^2+1}$ without using arctangent.

Discussion: This is an iconic residue-theorem computation, and is (also) useful in practice. First, basically from the definition of path integral, the calculus-style integral is equal to the corresponding integral over the real line, and we acknowledge this by using variable z instead of the usually-real variable x:

$$\int_{-\infty}^{\infty}\frac{dx}{1+x^2} \;=\; \int_{-\infty}^{\infty}\frac{dz}{1+z^2}$$

The parametrization can be by the real line itself, by $t\to t$ for $t\in\mathbb{R}$. As usual, infinite integrals are limits of finite ones, so these integrals are

$$\lim_{R\to\infty}\int_{-R}^{R}\frac{dz}{1+z^2}$$

As usual, we want to close up the path of integration by adding an *auxiliary arc*, depending on R, which as $R\to+\infty$ goes to 0. Here, the auxiliary half-circle arc α_R parametrized by $t\to Re^{it}$ for $t\in[0,2\pi]$ closes up the integral on $[-R,R]$, to a closed path γ_R. By residues, for any $R>1$ (so that the pole of $1/(1+z^2)$ at $z=i$ is inside the path), since there are no poles inside γ_R except i,

$$\int_{\gamma_R}\frac{dz}{1+z^2} \;=\; 2\pi i\mathrm{Res}_{z=i}\frac{1}{1+z^2} \;=\; 2\pi i\mathrm{Res}_{z=i}\frac{1}{z-i}\cdot\frac{1}{z+i}$$

At this point, it is good to observe a fairly general useful way to evaluate residues: for f holomorphic on a neighborhood of z_o, the residue of $f(z)/(z-z_o)$ at z_o is $f(z_o)$. This is proven by obtaining the Laurent series of $f(z)/(z-z_o)$ at z_o by dividing the power series of $f(z)$ at z_o by $z-z_o$. Thus,

$$\int_{\gamma_R}\frac{dz}{1+z^2} \;=\; 2\pi i\frac{1}{i+i} \;=\; \pi$$

The trivial estimate on the integral over the auxiliary arc gives

$$\Big|\int_{\alpha_R} \frac{dz}{(z-i)(z+i)}\Big| \;=\; \text{length} \times \text{maximum on the arc} \;\le\; \pi R \times \frac{1}{(R-1)^2} \;\longrightarrow\; 0 \qquad (\text{as } R \to +\infty)$$

That is, as desired, in the limit, the integral over the auxiliary arc goes to 0. Combining all this,

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+1} \;=\; \pi$$

as desired. ///

[02.10] Compute $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^4+1}$

Discussion: As in the previous example, the infinite integral is a limit of finite limits

$$\int_{-\infty}^{\infty} \frac{dx}{x^4+1} \;=\; \lim_{R\to+\infty} \int_{-R}^{R} \frac{dx}{x^4+1}$$

The denominator has zeros at eighth roots of unity, namely, $\zeta = \zeta_8 = e^{\pi i/4}$, $\zeta^3 = e^{3\pi i/4}$, $\zeta^5 = e^{5\pi i/4}$, $\zeta^7 = e^{7\pi i/4}$. Let γ_R be the path from $-R$ to R along the real line, and then along the auxiliary arc, the circle of radius R in the upper half-plane, from $+R$ back to $-R$. The integral over the arc is estimated via the trivial estimate:

$$\Big|\int_{\text{arc } R} \frac{dx}{x^4+1}\Big| \le \text{length}(\text{arc } R) \cdot \sup_{\text{on arc } R} \Big|\frac{1}{z^4+1}\Big| \le \pi R \cdot \frac{1}{(R-1)^4}$$

This goes to 0 as $R \to +\infty$. Thus, using the Residue Theorem, the original integral is

$$\int_{-\infty}^{\infty} \frac{dx}{x^4+1} \;=\; \lim_R \int_{\gamma_R} \frac{dz}{1+z^4} \;=\; \lim_R 2\pi i \,\mathrm{Res}_{z=\zeta,\,\zeta^3} \frac{1}{1+z^4}$$

$$=\; 2\pi i\Big(\frac{1}{(\zeta-\zeta^3)(\zeta-\zeta^5)(\zeta-\zeta^7)} + \frac{1}{(\zeta^3-\zeta)(\zeta^3-\zeta^5)(\zeta^3-\zeta^7)}\Big)$$

recalling the convenient fact that the residue at z_o of $g(z)/(z-z_o)$ for g holomorphic at z_o is $g(z_o)$. This is

$$2\pi i\Big(\frac{1}{(\sqrt{2})(2\zeta)(i\sqrt{2})} + \frac{1}{(-\sqrt{2})(i\sqrt{2})(2i\zeta)}\Big) \;=\; \frac{\pi i}{2}\Big(\frac{1}{i\zeta} + \frac{1}{\zeta}\Big) \;=\; \frac{\pi\zeta^2}{2} \cdot \frac{1-i}{\zeta} \;=\; \frac{\pi}{2} \cdot \frac{1+i}{\sqrt{2}} \cdot (1-i) \;=\; \frac{\pi}{\sqrt{2}}$$

as required. ///

[02.11] Compute $\displaystyle\int_{-\infty}^{\infty} \frac{x\,dx}{x^4+1}$

Discussion: This example illustrates the relevance of *symmetry*. That is, the function $1/(x^4+1)$ is *even*, and the function x is *odd*, so $x/(x^4+1)$ is *odd*. But then the integral over the whole real line is equal to its own negative, under the change of variables $x \to -x$, so is 0. ///

[02.12] Compute $\displaystyle\int_{-\infty}^{\infty} \frac{x^2\,dx}{x^4+1}$

Discussion: By residues, using a semi-circle in the upper (or lower) half-plane as auxiliary arc. Picks up residues at the two primitive 8^{th} roots of unity in the upper (or lower) half-plane... ///

[02.13] Compute $\int_{-\infty}^{\infty} \frac{e^{itx}\,dx}{x^2+1}$ with real t.

Discussion: By residues, depending on the sign of $t \in \mathbb{R}$, using a semi-circle in the upper/lower half-plane. The choice depends on the sign of t, so that e^{itz} is *bounded* in z in the corresponding half-plane.

As usual, the integral is equal to a contour/path integral, whose sense we emphasize by using z as a dummy variable instead of x.

As usual, the infinite integral is a limit of integrals $\int_{-R}^{R}$ as $R \to +\infty$. For $t \geq 0$, $z \to e^{itz}$ is *bounded* in $\mathrm{Im}(z) \geq 0$, because for real t and $z = x + iy$,

$$|e^{itz}| \;=\; e^{\mathrm{Re}\,(itz)} \;=\; e^{-ty}$$

Thus, the trivial estimate of the integral on the auxiliar arc α_R (the half-circle in the upper half-plane) is that it's dominated by

$$\text{length } \alpha_R \times \text{max of integrand on arc} \;\leq\; \pi R \times \frac{1}{(R-1)^2} \;\longrightarrow\; 0 \qquad\qquad (\text{as } R \to +\infty)$$

Thus, for $t \geq 0$, by residues, the integral is essentially the sum of residues of the integrand in the upper half-plane: it is

$$2\pi i \cdot \mathrm{Res}_{z=i} \frac{e^{itz}}{z^2+1} \;=\; 2\pi i \cdot \frac{e^{it(i)}}{i+i} \;=\; \pi \cdot e^{-t}$$

using the general fact that $\mathrm{Res}_{z=z_o} f(z)/(z-z_o) = f(z_o)$ for f holomorphic at z_o.

For $t \leq 0$, the auxiliary arce is a semi-circle in the lower half-plane. Because the closed path is traced in a mathematically negative direction, the *negative* of $2\pi i$ times the residues is picked up, giving

$$-2\pi i \mathrm{Res}_{z=-i} \frac{e^{itz}}{z^2+1} \;=\; -2\pi i \frac{e^{it(-i)}}{(-i)-i} \;=\; \pi \cdot e^{t}$$

We can express both simultaneously as $\pi e^{-|t|}$. ///

[02.14] Compute $\int_{-\infty}^{\infty} \frac{\sin(tx)\,dx}{x^2+1}$ with real t.

Discussion: Another short-circuited issue: the integrand is *odd*, so its integral over the whole real line (stable under $x \to -x$) is 0, *by pure thought.* ///

[02.15] Compute $\int_{-\infty}^{\infty} \frac{\cos(tx)\,dx}{x^2+1}$ with real t.

Discussion: Use $\cos(tx) = \frac{e^{itx}+e^{-itx}}{2}$, and for auxiliary arcs use a semi-circle in the upper half-plane for one term, and in the lower half-plane for the other, also depending on the sign of t. This reduces to a previous example. In fact, since $1/(x^2+1)$ is *even*, and $\cos(tx)$ is the average of $e^{\pm itx}$, the outcome is the same as with e^{itx} in place of $\cos(tx)$, namely, $\pi e^{-|t|}$. ///

[02.16] Compute $\int_{-\infty}^{\infty} \frac{e^{itx}\,dx}{(x+i)^2}$ with real t.

Discussion: As usual, to compute by residues, we closed up the paths $[-R, R]$ by auxiliary arcs. As in other example, the choice of auxiliary arc depends on the sign of $t \in \mathbb{R}$: for $t \geq 0$, $z \to e^{itz}$ is bounded in the *upper* half-plane, so we use an upper half-plane semi-circle (at 0) of radius R, while for $t \leq 0$, the exponential is bounded in the *lower* half-plane, and we use a lower half-plane semi-circle of radius R, traversing a closed curve in the negative direction.

For $t \geq 0$, the trivial/easy estimate on the R^{th} auxiliary arc is that it is dominated by

$$\text{length}x\text{maximum absolute value of integrand} \;\leq\; \pi R \times \frac{1}{(R-1)^2} \;\longrightarrow\; 0 \qquad\qquad (\text{as } R \to +\infty)$$

Thus, in this case, for $R \geq 2$, the integral can be evaluated by residues, but/and there are no singularities of $e^{itz}/(z^2+1)$ in the upper half-plane, so producing 0.

For $t \leq 0$, the trivial/easy estimate on the R^{th} auxiliary arc again shows that it goes to 0, so theintegral can be evaluated by residues. The point $z = -i$ is the only singularity (in the lower half-plane), and the integral is (integrating in the negative direction)

$$-2\pi i \times \text{Res}_{z=-i} \frac{e^{itz}}{(z-(-i))^2} \;=\; -2\pi i \times \frac{d}{dz} e^{itz}\Big|_{z=-i}$$

by the somewhat general formula

$$\text{Res}_{z=z_o} \frac{f(z)}{(z-z_o)^n} \;=\; \frac{f^{(n-1)}(z_o)}{(n-1)!}$$

This gives

$$-2\pi i \cdot (it) \cdot e^{it(-i)} \;=\; 2\pi t \cdot e^t$$

Thus, in a form that perhaps emphasizes the qualitative features, the integral is

$$\begin{cases} 0 & (\text{for } t \geq 0) \\ 2\pi |t| \cdot e^{-|t|} & (\text{for } t \leq 0) \end{cases}$$

[02.17] Compute $\displaystyle\int_{-\infty}^{\infty} \frac{e^{itx}\, dx}{(x+i)^{100}}$ with real t.

Discussion: The same argument as in the previous example gives 0 for $t \geq 0$ and for $t \leq 0$ it is

$$-2\pi i \cdot (it)^{n-1} \cdot (n-1)! \cdot e^{it(-i)} \;=\; -2\pi \cdot i^n \cdot t^{n-1} \cdot (n-1)! \cdot e^{-|t|}$$

as required. ///

[02.18] Compute $\displaystyle\int_0^{\infty} \frac{x\, dx}{1+x^3}$

Discussion: As usual, the integral is the limit of finite integrals $\int_0^R$ as $R \to +\infty$. Let γ_R be the path from 0 to R along the real line, then counter-clockwise along the circle of radius R to $R \cdot e^{2\pi i/3}$, then back along the straight line to 0. This path is chosen because the integral from $R \cdot e^{2\pi i/3}$ to 0 is very simply related to the original:

$$\int_R^0 \frac{(e^{2\pi i/3}t)\, d(e^{2\pi i/3}t)}{1+(e^{2\pi i/3}t)^3} \;=\; -e^{4\pi i/3} \int_0^R \frac{t\, dt}{1+t^3}$$

The integral along the arc is easily estimate by the trivial estimate:

$$\Big|\int_{\text{arc } R} \frac{z\, dz}{1+z^3}\Big| \;\leq\; \text{length}(\text{arc } R) \cdot \sup_{\text{on arc } R} \Big|\frac{z}{1+z^3}\Big| \;\leq\; \frac{2\pi R}{3} \cdot \frac{R}{(R-1)^3}$$

which goes to 0 as $R \to +\infty$. The integral over γ_R can be evaluated by residues: for $R > 1$, there is a single singularity inside γ_R, at the sixth root of unity $\zeta = \zeta_6 = e^{\pi i/3}$. Noting that

$$z^3 + 1 \;=\; (z+1)(z^2 - z + 1) \;=\; (z+1)(z-\zeta)(z-\zeta^{-1})$$

and that $-e^{4\pi i/3} = \zeta$, we have

$$(1+\zeta)\int_0^\infty \frac{x\;dx}{1+x^3} \;=\; \lim_R \int_{\gamma_R} \frac{z\;dz}{1+z^3} \;=\; 2\pi i\;\mathrm{Res}_{z=\zeta}\frac{z}{1+z^3} \;=\; 2\pi i\frac{\zeta}{(\zeta+1)(\zeta-\zeta^{-1})}$$

so

$$\int_0^\infty \frac{x\;dx}{1+x^3} \;=\; 2\pi i\frac{\zeta}{(\zeta+1)^2\,(\zeta-\zeta^{-1})} \;=\; 2\pi i\frac{1}{(\zeta+1)(\zeta^{-1}+1)\,(i\sqrt{3})} \;=\; \frac{2\pi}{(\frac{1+i\sqrt{3}}{2}+1)(\frac{1-i\sqrt{3}}{2}+1)\sqrt{3}}$$

$$=\; \frac{2\pi}{(\frac{9}{4}+\frac{3}{4})\sqrt{3}} \;=\; \frac{2\pi}{3\sqrt{3}}$$

as required. ///

[02.19] Compute $\displaystyle\int_0^\infty \frac{x^{1/4}\;dx}{1+x^2}$

Discussion: We can replace x by x^4, and reduce to some previous examples, *or* look at this as a special case of

$$\int_0^\infty \frac{x^s\;dx}{1+x^2} \qquad\qquad (\text{with } -1 < \mathrm{Re}(s) < 1)$$

The latter is amenable to use of the *keyhole* or *Hankel* contour... *[... iou ...]* ///

[02.20] Compute $\displaystyle\frac{1}{1} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \ldots$

Discussion: Arrange to evaluate the infinite sum by residues, by using the function $2\pi i/(e^{2\pi iz}-1)$, which we will check has simple poles with residues 1 at integers, and for z bounded away from integers is *bounded.* Granting that for a moment, letting γ_T be a counter-clockwise path around the square with vertices $\pm T \pm iT$ with $T \in \frac{1}{2} + \mathbb{Z}$, by residues

$$\int_{\gamma_T} \frac{2\pi i}{e^{2\pi iz}-1}\cdot\frac{1}{z^4}\;dz \;=\; \sum_{0\le|n|<T} \mathrm{Res}_{z=n}\frac{2\pi i}{e^{2\pi iz}-1}\cdot\frac{1}{z^4} \;=\; \sum_{0<|n|<T}\frac{1}{n^4} + \mathrm{Res}_{z=0}\frac{2\pi i}{e^{2\pi iz}-1}\cdot\frac{1}{z^4}$$

Because of the division by z^4, the latter residue is visibly the coefficient of z^3 in the Laurent expansion of $2\pi i/(e^{2\pi iz}-1)$, which is determined by expanding $1/(e^z-1)$

$$\frac{1}{e^z-1} \;=\; \frac{1}{(1+z+\frac{z^2}{2}+\frac{z^3}{6}+\frac{z^4}{24}+\ldots)-1} \;=\; \frac{1}{z+\frac{z^2}{2}+\frac{z^3}{6}+\frac{z^4}{24}+\ldots}$$

$$=\; \frac{1}{z}\cdot\frac{1}{1+\Big(\frac{z}{2}+\frac{z^2}{6}+\frac{z^3}{24}+\ldots\Big)} \;=\; \frac{1}{z}\Big(1-\Big(\frac{z}{2}+\frac{z^2}{6}+\ldots\Big)+\Big(\frac{z}{2}+\frac{z^2}{6}+\ldots\Big)^2-\Big(\frac{z}{2}+\frac{z^2}{6}+\ldots\Big)^3+\ldots\Big)$$

The z^3 coefficient of the latter is

$$-\frac{1}{5!}+\Big(2\cdot\frac{1}{2!}\cdot\frac{1}{4!}+1\cdot\Big(\frac{1}{3!}\Big)^2\Big)-3\cdot\Big(\frac{1}{2!}\Big)^2\cdot\frac{1}{3!}+\Big(\frac{1}{2!}\Big)^4 \;=\; -\frac{1}{120}+\frac{1}{24}+\frac{1}{36}-\frac{1}{8}+\frac{1}{16}$$

Replacing z by $2\pi iz$ in that Laurent expansion, and multiplying from the $2\pi i$ from the numerator multiplies this by $(2\pi i)^4 = 16\pi^4$, giving π^4 times

$$-\frac{16}{120}+\frac{16}{24}+\frac{16}{36}-\frac{16}{8}+\frac{16}{16} \;=\; -\frac{2}{15}+\frac{2}{3}+\frac{4}{9}-1 \;=\; \frac{-6+30+20-45}{45} \;=\; -\frac{1}{45}$$

so $-\pi^4/45$. Thus, still granting that everything works out, we have

$$\int_{\gamma_T} \frac{2\pi i}{e^{2\pi iz}}\,\frac{1}{z^4}\,dz \;=\; \sum_{0<|n|<T}\frac{1}{n^4}+\mathrm{Res}_{z=0}\frac{2\pi i}{e^{2\pi iz}-1}\cdot\frac{1}{z^4} \;=\; \sum_{0<|n|<T}\frac{1}{n^4}-\frac{\pi^4}{45}$$

Taking the limit, the integral goes to 0, so

$$0 \;=\; \lim_T \sum_{0<|n|<T}\frac{1}{n^4}-\frac{\pi^4}{45} \;=\; 2\cdot\sum_{n\ge 1}\frac{1}{n^4} \;-\; \frac{\pi^4}{45}$$

giving the claimed result. For the other details:

The function $2\pi i/(e^{2\pi iz}-1)$ has no singularities unless the denominator is 0, which occurs exactly at integers. It is $\mathbb{Z}$-periodic, so to check that its residue at 0 is 1: as above,

$$\frac{2\pi i}{e^{2\pi iz}-1} \;=\; \frac{2\pi i}{(1+(2\pi iz)+\frac{(2\pi iz)^2}{2}+\ldots)-1} \;=\; \frac{2\pi i}{2\pi iz+\frac{(2\pi iz)^2}{2}+\ldots} \;=\; \frac{1}{z+\frac{2\pi iz^2}{2}+\ldots}$$
$$=\; \frac{1}{z}\cdot\frac{1}{1+\Big(2\pi i\frac{z}{2}+\ldots\Big)} \;=\; \frac{1}{z}\Big(1-\Big(\frac{2\pi iz}{2}+\ldots\Big)+\ldots\Big)$$

To check that this function is *bounded* for z away from integers, first observe that $|e^{2\pi iz}|\le\frac{1}{e}$ for $\mathrm{Im}(z)\ge\frac{1}{2\pi}$, and $|e^{2\pi iz}|\ge e$ for $\mathrm{Im}(z)\le-\frac{1}{2\pi}$. In both cases, $e^{2\pi iz}-1$ is bounded away from zero, so $2\pi i/(e^{2\pi iz}-1)$ is bounded.

For $|\mathrm{Im}(z)|\le\frac{1}{2\pi}$, again use periodicity, to reduce to the set where $|\mathrm{Im}(z)|\le\frac{1}{2\pi}$, $0\le\mathrm{Re}(z)\le 1$, and $|z-0|\ge\frac{1}{2}$ and $|z-1|\ge 1$. This set is *compact*, and $|2\pi i/(e^{2\pi iz}-1)|$ is continuous on it, so is bounded. This completes the checking of the background details to make things work. ///

[02.21] Show that z^n+z-1 has n zeros inside the circle $|z|=2$.

Discussion: Implicitly, probably $n\ge 3$ or something of this sort. This is about Rouché's theorem, or possibly a simpler argument, since the question only involves polynomials. An orthodox approach says that $f(z)=z^n-1$ and $g(z)=z^n+z-1$ have the same number of roots inside $|z|=2$ if $|f(z)-g(z)|<|f(z)|$ on that circle. As designed,

$$|f(z)-g(z)| \;=\; |z| \;=\; 2 \;<\; 2^n-1 \;\le\; |z^n-1| \;=\; |f(z)| \qquad\qquad (\text{on } |z|=2)$$

so that Rouché's theorem is applicable. ///

[02.22] Are there complex zeros of $z-\cos z$ beyond the obvious one on $\mathbb{R}$?

Discussion: This is about Rouché's theorem, but not so easy to definitively address. ///

[02.23] For a bounded sequence of complex numbers c_n, prove that $\displaystyle\sum_{n=0}^{\infty} c_n\frac{z^n}{z^n+1}$ converges to a holomorphic function on $|z|<1$.

Discussion: Each summand is holomorphic on $|z| < 1$, because of the *quotient rule*, and that the numerator and denominator are polynomials, hence holomorphic.

To prove that the sum $\sum_n f_n$ of a sequence of holomorphic functions on $|z| < 1$ is itself holomorphic, it suffices to prove that the convergence is *uniform on compacts.* The compact subsets of the open disk are all contained in compact disks $|z| \le r$ for $r < 1$, so it suffices to consider just those sets $|z| \le r$.

Given $r < 1$, there is large-enough N such that $r^n \le \frac{1}{2}$ for all $n \ge N$, for example taking $N \ge \frac{\log \frac{1}{2}}{\log r}$. For $|z| \le r$ and $n \ge N$,

$$\Big|\frac{z^n}{1+z^n}\Big| \;\le\; \frac{|z|^n}{1-\frac{1}{2}} \;\le\; 2r^n$$

Thus, given $0 < r < 1$, let N so that $r^n \le \frac{1}{2}$ for all $n \ge N$. Given $\varepsilon > 0$, for $m, n \ge N$, with $|c_n| \le B$ for all n,

$$\Big|\sum_{m\le j<n} c_n\,\frac{z^j}{1+z^j}\Big| \;\le\; B\cdot\sum_{m\le j<n} 2r^j \;<\; B\cdot\sum_{m\le j<\infty} 2r^j \;\le\; B\frac{2r^m}{1-r}$$

Increasing N if necessary, this is smaller than ε. ///

There are several viable variant approaches. Among others: expanding the power series for each $z^n/(z^n+1)$, although one should be careful *not* to suggest that a sum of holomorphic functions on a disk is holomorphic on that disk, since $\sum_n c_n\, z^n$ can have arbitrary radius of convergence, while the summands $c_n\, z^n$ have infinite radius of convergence. Invocation of Morera's theorem also works here. ///

[02.24] Let f be a continuous, bounded real-valued function on $\mathbb{R}$, extending to a bounded, holomorphic function on the upper half-plane $\mathfrak{H}$. Show f is constant.

Discussion: This is a combination of the Reflection Principle and Liouville's Theorem. Namely, by the Reflection Principle, defining f in the lower half-plane by $f(z) = \overline{f(\overline{z})}$ gives a holomorphic function on $\mathbb{C}$. This extension is still bounded, so by Liouville is constant. ///

[02.25] Evaluate the Fourier transform $\displaystyle\int_{-\infty}^{\infty} e^{-itx}\cdot\frac{1}{(x+i)^s}\,dx$ for complex s with $\mathrm{Re}(s) > 1$, using the Gamma function.

Discussion: My preferred approach to this, while not the shortest, illustrates some important methodological and technical points.

Recall that the Identity Principle gives

$$\int_0^\infty e^{-uz}\,u^s\,\frac{du}{u} \;=\; z^{-s}\,\Gamma(s) \qquad\qquad (\text{for } \mathrm{Re}(z) > 0 \text{ and } \mathrm{Re}(s) > 0)$$

Using this identity in the problem at hand,

$$\int_{-\infty}^\infty e^{-itx}\,\frac{1}{(x+i)^s}\,dx \;=\; i^{-s}\int_{-\infty}^\infty e^{-itx}\,\frac{1}{(1-ix)^s}\,dx \;=\; i^{-s}\frac{1}{\Gamma(s)}\int_{-\infty}^\infty\int_0^\infty e^{-itx}\,e^{-u(1-ix)}u^s\,\frac{du}{u}\,dx$$

Changing the order of integration, *if justifiable*, would give

$$i^{-s}\frac{1}{\Gamma(s)}\int_0^\infty e^{-u}\,\Big(\int_{-\infty}^\infty e^{i(u-t)x}\,dx\Big)u^s\,\frac{du}{u}$$

The difficulty is that the inner integral is not at all convergent in a classical, pointwise sense. Thus, with hindsight, the interchange of integrals is not justifiable in classical terms.

Nevertheless, that integral should remind us of *Fourier Inversion*: for nice-enough functions,

$$f(x) \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi x}\Big(\int_{-\infty}^{\infty} e^{-i\xi u}\; f(u)\;du\Big)\;d\xi$$

In particular, there is an illuminating heuristic, or near-proof, for Fourier Inversion, involving the same not-classically-justifiable interchange of integrals:

$$\int_{-\infty}^{\infty} e^{i\xi x}\Big(\int_{-\infty}^{\infty} e^{-i\xi u}\; f(u)\;du\Big)\;d\xi \;=\; \int_{-\infty}^{\infty}\Big(\int_{-\infty}^{\infty} e^{i\xi(x-u)}\;d\xi\Big)f(u)\;du \qquad (?????)$$

Since we *know* that this should be $2\pi\cdot f(x)$, it must be that, in effect,

$$\int_{-\infty}^{\infty} e^{i\xi(x-u)}\;d\xi \;=\; 2\pi\cdot\delta(x-u) \qquad \text{(Dirac delta)}$$

Granting this heuristic for a moment, the integral at hand would become

$$2\pi\cdot i^{-s}\frac{1}{\Gamma(s)}\int_{0}^{\infty} e^{-u}\;\delta(u-t)\;u^s\;\frac{du}{u} \;=\; \begin{cases} \dfrac{2\pi}{i^s\,\Gamma(s)}\;e^{-t}\;t^{s-1} & (\text{for } t\ge 0)\\[2mm] 0 & (\text{for } t<0)\end{cases}$$

In our context this is only a *heuristic*, but it *suggests* the correct value for the integral, and we can attempt to *check* the outcome of the heuristic, via Fourier Inversion. Thus, disregarding the constant $2\pi/i^s\Gamma(s)$ for a moment, compute the inverse Fourier transform of

$$F(t) \;=\; \begin{cases} e^{-t}\;t^{s-1} & (\text{for } t\ge 0)\\[2mm] 0 & (\text{for } t<0)\end{cases}$$

This is

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi x}\;F(\xi)\;d\xi \;=\; \frac{1}{2\pi}\int_{0}^{\infty} e^{i\xi x}\;e^{-\xi}\;\xi^{s-1}\;d\xi \;=\; \frac{1}{2\pi}\int_{0}^{\infty} e^{i\xi x}\;e^{-\xi}\;\xi^{s}\;\frac{d\xi}{\xi}$$

$$=\; \frac{1}{2\pi}\int_{0}^{\infty} e^{-\xi(1-ix)}\;\xi^{s}\;\frac{d\xi}{\xi} \;=\; \frac{1}{2\pi}\frac{1}{(1-ix)^s}\int_{0}^{\infty} e^{-\xi}\;\xi^{s}\;\frac{d\xi}{\xi} \;=\; \frac{1}{2\pi}\frac{1}{(1-ix)^s}\Gamma(s) \;=\; \frac{1}{2\pi}\;i^s\;\frac{1}{(x+i)^s}\Gamma(s)$$

by the same identity. Thus, all the constants correctly cancel, and by Fourier Inversion we see that the heuristic gave the true outcome:

$$\int_{-\infty}^{\infty} e^{-itx}\;\frac{1}{(x+i)^s}\;dx \;=\; \begin{cases} \dfrac{2\pi}{i^s\,\Gamma(s)}\;e^{-t}\;t^{s-1} & (\text{for } t\ge 0)\\[2mm] 0 & (\text{for } t<0)\end{cases}$$

as desired. ///

[02.26] Show that $f(z)=\displaystyle\int_0^1 \frac{dt}{t\cdot z+(1-t)\cdot z_o}$ is holomorphic at any z_1 such that 0 is *not* on the straight line segment with endpoints z_o and z_1. Find the radius of convergence of its power series expanded at $z_o=-4+3i$.

Discussion: As with the case $z_o=1$, holomorphy is proven via Morera's Theorem, for example.

For any z_o such that the line segment connecting z_o and $-4+3i$ does not pass through 0, the corresponding function is holomorphic at $-4+3i$, so admits a power series expansion there. From Cauchy theory, this power series will converge on the largest open disk centered at $-4+3i$ on which there is a holomorphic function agreeing with $f(z)$.

Because of the potential blow-up of the integral, not to mention knowing that $\log 0$ cannot have a value making the function holomorphic, no one of these functions $f(z)$ can be holomorphic at 0, so 0 is not contained in any disk on which $f(z)$ is holomorphic. We show that there is no *other* obstacle.

The functions $f(z)$ defined via different z_o only differ by constants, the value of the integral of $1/w$ from one z_o to another. Thus, in particular, we could consider $z_o = -4+3i$ without loss of generality, in the sense that if we find radius of convergence equal to the distance to 0 (namely, 5), then, since we cannot do any *better*, we're done.

The function $f(z)$ defined with $z_o = -4+3i$ is holomorphic on the slit plane obtained by removing from $\mathbb{C}$ the ray from 0 passing through $-(-4+3i)$. The largest disk centered at $-4+3i$ in this half-plane indeed has radius 5, the distance from $-4+3i$ to 0. ///
