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Complex analysis examples discussion 03

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[03.1] Adapt the reflection principle to show that a holomorphic function on the unit disk, extending to a continuous function on the closed unit disk, with $|f(z)| = 1$ on the unit circle, extends to a holomorphic function on $\mathbb{C}$ except for finitely-many poles. (Hint: for example, $z \to \frac{z-i}{-iz+1}$ maps the real line to the unit circle.)

Discussion: The inverse Cayley map $C^{-1}(z) = \frac{z-i}{-i+z}$ maps the upper half-plane to the disk, and the real line to the unit circle. Thus, $F = C^{-1} \circ f \circ C$ is holomorphic on the upper half-plane taking values in the upper half-plane, extending to a real-valued continuous function on $\mathbb{R}$, satisfies the hypotheses of the most standard reflection principle. Thus, by the reflection principle, F extends by

$$F(\overline{z}) \;=\; \overline{F(z)} \qquad\qquad (\text{for } z \in \mathfrak{H})$$

Let α be the complex conjugation map. The Cayley map interacts in a coherent way with conjugation: $C \circ \alpha = \alpha \circ C^{-1}$, and $C^{-1} \circ \alpha = \alpha \circ C^{-1}$.

Letting F still denote the extension,

$$f \;=\; C^{-1} \circ F \circ C \;=\; C^{-1} \circ (\alpha \circ F \circ \alpha) \circ C$$

by using the formula for the extension of F to the lower half-plane. Now rearrange to rewrite the latter expression in terms of f itself. First, the interaction of α and C gives

$$\alpha \circ C \circ F \circ C^{-1} \circ \alpha \;=\; \alpha \circ C^2 \circ (C^{-1} \circ F \circ C) \circ C^{-2} \circ \alpha \;=\; \alpha \circ C^2 \circ f \circ C^{-2} \circ \alpha$$

The Cayley map has the important property that $C^2(z) = C^{-2}(z) = 1/z$, so this gives

$$f(z) \;=\; \alpha \circ \frac{1}{f(1/\overline{z})} \;=\; 1/\overline{f(1/\overline{z})}$$

This is the desired reflection formula for the circle. Note that *zeros* of f give rise to *poles* of the reflected part. The hypotheses on f in the disk assure that it has finitely-many zeros there, so the reflected part has finitely-many poles. ///

[03.2] Show that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

Discussion: First, a change of variables

$$\Gamma(\tfrac{1}{2}) \;=\; \int_0^\infty e^{-t} t^{\frac{1}{2}} \frac{dt}{t} \;=\; 2\int_0^\infty e^{-t^2} t^1 \frac{dt}{t} \;=\; \int_{\mathbb{R}} e^{-t^2}\, dt$$

and then the standard calculus device: squaring and converting to polar coordinates:

$$\Big(\int_{\mathbb{R}} e^{-t^2}\, dt\Big)^2 \;=\; \int_{\mathbb{R}^2} e^{-(x^2+y^2)}\, dx\, dy \;=\; \int_0^{2\pi}\int_0^\infty e^{-r^2}\, r\, dr\, d\theta \;=\; 2\pi \int_0^\infty e^{-r^2}\, r\, dr$$

$$=\; \pi \int_0^\infty e^{-r^2}\, 2r\, dr \;=\; \pi \int_0^\infty e^{-r}\, dr \;=\; \pi$$

Thus, $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. ///

Alternatively, the functional equation $\Gamma(s)\Gamma(1-s) = \pi/\sin\pi s$ gives

$$\Gamma(\tfrac{1}{2})^2 \;=\; \Gamma(\tfrac{1}{2})\cdot\Gamma(1-\tfrac{1}{2}) \;=\; \frac{\pi}{\sin\frac{\pi}{2}} \;=\; \pi$$

and we reach the same conclusion. ///

[03.3] Show that $\overline{\Gamma(s)} = \Gamma(\overline{s})$ for all $s\in\mathbb{C}$.

Discussion: This is an instance of application of the Identity Principle. Namely, from its integral representation, $\Gamma(s)$ is $\mathbb{R}$-valued for $s\in(0,+\infty)$. Thus, for $s\in(0,+\infty)$, $\Gamma(s)=\overline{\Gamma(\overline{s})}$. Thus, $s\to\overline{\Gamma(\overline{s})}$ is holomorphic. By the Identity Principle, we have the asserted equality everywhere.

We might want to recall the computation that for f holomorphic $z\to\overline{f(\overline{z})}$ is again holomorphic, by checking complex differentiability:

$$\frac{\overline{f(\overline{z+h})}-\overline{f(\overline{z})}}{h} \;=\; \overline{\Big(\frac{f(\overline{z}+\overline{h})-f(\overline{z})}{\overline{h}}\Big)}$$

Since $h\to 0$ is equivalent to $\overline{h}\to 0$, the limit as $\overline{h}\to 0$ of the expression under the complex conjugation on the right-hand side is $f'(\overline{z})$. In particular, the limit exists. Thus, $z\to\overline{f(\overline{z})}$ is holomorphic. ///

[03.4] Show that $|\Gamma(\frac{1}{2}+it)| = \sqrt{\dfrac{\pi}{\cosh\pi t}}$ for real t. (Thus, in contrast to the horizontal super-exponential *growth* of $n\to n!$, the vertical behavior is exponential decrease.)

Discussion: Use the functional equation and $\overline{\Gamma(s)}=\Gamma(\overline{s})$:

$$|\Gamma(\tfrac{1}{2}+it)|^2 \;=\; \Gamma(\tfrac{1}{2}+it)\cdot\overline{\Gamma(\tfrac{1}{2}+it)} \;=\; \Gamma(\tfrac{1}{2}+it)\cdot\Gamma(\overline{\tfrac{1}{2}+it}) \;=\; \Gamma(\tfrac{1}{2}+it)\cdot\Gamma(\tfrac{1}{2}-it)$$

$$=\; \Gamma(\tfrac{1}{2}+it)\cdot\Gamma(1-(\tfrac{1}{2}+it)) \;=\; \frac{\pi}{\sin\pi(\frac{1}{2}+it)} \;=\; \frac{2\pi i}{e^{\pi i(\frac{1}{2}+it)}-e^{-\pi i(\frac{1}{2}+it)}} \;=\; \frac{2\pi i}{ie^{-\pi t}+ie^{\pi t}} \;=\; \frac{\pi}{\cosh\pi t}$$

Thus,

$$|\Gamma(\tfrac{1}{2}+it)| \;=\; \sqrt{\frac{\pi}{\cosh\pi t}}$$

as claimed. ///

[03.5] Prove that $f(z)=\displaystyle\int_0^1 \frac{e^{tz}\,dt}{t^2+1}$ is holomorphic.

Discussion: The simplest argument might be to invoke Morera's theorem after changing order of integration. The change of order is easily justifiable, since one is looking at a continuous function of two variables. That is, for each $t\in[0,1]$, the function $z\to\frac{e^{tz}}{t^2+1}$ is holomorphic, and the function of two variables is continuous. Thus, letting γ be a small triangle,

$$\int_\gamma\int_0^1 \frac{e^{tz}\,dt}{t^2+1}\,dz \;=\; \int_0^1\int_\gamma \frac{e^{tz}}{t^2+1}\,dz\,dt \;=\; \int_0^1 0\,dt \;=\; 0$$

by applying Cauchy's theorem to $z\to\frac{e^{tz}}{t^2+1}$. By Morera, $f(z)$ is continuous. ///

Another approach is to view the integral as a uniform limit of a sequence of finite (Riemann) sums, each of which is holomorphic, being a finite sum of holomorphic functions, and then invoke the holomorphy of uniform (on compacts) limits of holomorphic functions.

[03.6] Prove that $f(z) = \int_0^\infty \frac{e^{-tz}\,dt}{t^2+1}$ is holomorphic for $\mathrm{Re}(z) > 0$.

Discussion: Using the previous example, it would suffice to show that the sequence of finite integrals

$$f_n(z) \;=\; \int_0^n \frac{e^{-tz}\,dt}{t^2+1}$$

converges uniformly to $f(z)$ for z in compact subsets of $\mathrm{Re}(z) > 0$, since these finite integrals are holomorphic functions, via Morera.

For fixed $\delta > 0$ and $\mathrm{Re}(z) \geq \delta$, for $N \leq m \leq n$,

$$\Big| f_m(z) - f_n(z) \Big| \;\leq\; \int_m^n \frac{e^{-t\delta}\,dt}{t^2+1} \;\leq\; \int_N^\infty e^{-t\delta}\,dt \;=\; \frac{e^{-N\delta}}{\delta}$$

This can be made smaller than a given $\delta > 0$ by taking N sufficiently large. ///

[03.7] Compute $\int_0^\infty \frac{x^s\,dx}{1+x^2}$

Discussion: The integral is absolutely convergent for $-1 < \mathrm{Re}(s) < 1$. Implicitly,

$$x^s \;=\; e^{s\log x}$$

where the logarithm is the one which is real-valued on $(0, +\infty)$. Use the Hankel/keyhole contour. First, the integral itself is a limit

$$\int_0^\infty \frac{x^s\,dx}{1+x^2} \;=\; \lim_{\varepsilon\to 0^+,\; R\to+\infty} \int_\varepsilon^R \frac{x^s\,dx}{1+x^2}$$

Let $H_{\varepsilon,R}$ be the Hankel/keyhole contour that comes from R along the real line to ε, then traces a circle of radius ε around 0 clockwise to ε, then back out to R. Let H_ε be the limiting case as $R \to +\infty$. We want the integral along that last part of the path, the outbound part from ε back out to R, to be the original integral $\int_\varepsilon^R x^s/(x^2+1)\,dx$. That is, we want the value of x^s to match.

On that small circle, the *argument* of x changes continuously, with a net decrease of 2π from its value on the in-bound part of the path. Requiring that x^s change *continuously* on that small circle, and be $e^{s\log x}$ with real-valued $\log x$ after traversing 2π radians counter-clockwise, requires that x^s be $e^{s(\log x + 2\pi i)}$ on the in-bound path. Thus,

$$\int_{\text{outbound+inbound}} \frac{x^2\,dx}{1+x^2} \;=\; (1 - e^{2\pi i s}) \int_\varepsilon^R \frac{x^2\,dx}{1+x^2}$$

Further, the main point of the keyhole trick is that, surprisingly, the limit over $\varepsilon \to 0^+$ is *reached in finite time*, in the sense that there is sufficiently small $\varepsilon_o > 0$ such that

$$\lim_{\varepsilon\to 0^+} \int_{H_{\varepsilon,R}} \frac{x^s\,dx}{1+x^2} \;=\; \int_{H_{\varepsilon_1,R}} \frac{x^s\,dx}{1+x^2} \qquad \text{(for all positive } \varepsilon_1 < \varepsilon_o\text{)}$$

Recall the proof: for $0 < \varepsilon_1 < \varepsilon_o$, let $\gamma_{\varepsilon_o,\varepsilon_1}$ be the closed path that traces counter-clockwise around the circle of radius ε_o from ε_o back to ε_o, then left to ε_1, then clockwise around a circle of radius ε_1 back to ε_1, then right to ε_o. In the interior of this path, the integrand is *holomorphic*. Adding the integral over $\gamma_{\varepsilon_o,\varepsilon_1}$ to the integral over $H_{\varepsilon_1,R}$ makes the integrals from ε_o to ε_1 (inbound) and from ε_1 to ε_o (outbound) cancel, and the integrals around the circles of radius ε_1 cancel, leaving $H_{\varepsilon_o,R}$. (Yes, one should draw a picture.)

To evaluate

$$\int_{H_{\varepsilon_1,R}} \frac{x^s\,dx}{1+x^2}$$

add an integral counter-clockwise around a circle σ_R of radius R from $R \in \mathbb{R}$ back to R. For $\mathrm{Re}(s) < 1$, the trivial estimate on this integral is

$$\Big|\int_{\sigma_R} \frac{x^s\,dx}{1+x^2}\Big| \;\le\; \text{length}\cdot\sup_{\sigma_R}\Big|\frac{x^s\,dx}{1+x^2}\Big| \;\le\; 2\pi R\cdot\frac{R^{\mathrm{Re}(s)}}{(R-1)^2} \;\longrightarrow\; 0 \qquad (\text{as } R\to+\infty, \text{ for } \mathrm{Re}(s)<1)$$

Thus,

$$\lim_R \int_{H_{\varepsilon_1,R}+\sigma_R} \frac{x^s\,dx}{1+x^2} \;=\; \int_{H_{\varepsilon_1}+\sigma_R} \frac{x^s\,dx}{1+x^2} \;=\; (1-e^{2\pi i s})\int_0^\infty \frac{x^s\,dx}{1+x^2}$$

On the other hand, the integral over the closed contour $H_{\varepsilon_1,R}+\sigma_R$ can be evaluated by *residues*: it is $2\pi i$ times the sum of residues in its interior. Inside that path, for small ε_1 and large R, there are exactly two poles, at $x = \pm i$, and both are simple. The value of $\arg x$ at i is obtained by moving counter-clockwise from the $\arg x = 0$ on $(0,+\infty)$, giving $\frac{\pi}{2}$. The argument at $-i$ is obtained by continuing counter-clockwise, giving $\frac{3\pi}{2}$. Thus,

$$\text{sum of residues} \;=\; \frac{e^{\frac{\pi i}{2}s}}{(-i)-i} + \frac{e^{\frac{3\pi i}{2}s}}{i-(-i)} \;=\; \frac{e^{\frac{\pi i}{2}s}}{-2i} + \frac{e^{\frac{3\pi i}{2}s}}{2i}$$

In summary,

$$\int_0^\infty \frac{x^s\,dx}{1+x^2} \;=\; \frac{1}{1-e^{2\pi i s}}\lim_R \int_{H_{\varepsilon_1}+\sigma_R} \frac{x^s\,dx}{1+x^2} \;=\; \frac{2\pi i}{1-e^{2\pi i s}}\Big(\frac{e^{\frac{\pi i}{2}s}}{-2i} + \frac{e^{\frac{3\pi i}{2}s}}{2i}\Big)$$

$$= \frac{\pi}{e^{2\pi i s}-1}\Big(e^{\frac{3\pi i}{2}s} - e^{\frac{\pi i}{2}s}\Big) \;=\; \pi\,\frac{e^{\frac{\pi i}{2}s} - e^{-\frac{\pi i}{2}s}}{e^{\pi i s} - e^{-\pi i s}} \;=\; \frac{\pi}{2}\,\frac{2}{e^{\frac{\pi i}{2}s} + e^{-\frac{\pi i}{2}s}} \;=\; \frac{\pi}{2\cos\frac{\pi s}{2}}$$

///

[03.8] Compute $\displaystyle\int_{-\infty}^\infty e^{-i\xi x}\, e^{-x^2}\,dx$

Discussion: The exponentials can be combined, and then complete the square:

$$-\tfrac{1}{2}i\xi\int_{-\infty}^\infty e^{-i\xi x}\,e^{-x^2}\,dx \;=\; \int_{-\infty}^\infty e^{-i\xi x}\,e^{-x^2}\,dx \;=\; \int_{-\infty}^\infty e^{-(x^2+i\xi x)}\,dx$$

$$= \int_{-\infty}^\infty e^{-(x^2+i\xi x-\frac{\xi^2}{4})-\frac{\xi^2}{4}}\,dx \;=\; e^{-\frac{\xi^2}{4}}\int_{-\infty}^\infty e^{-(x+\frac{i\xi}{2})^2}\,dx$$

The intuition at this point is that sliding the integral from $-\infty$ to $+\infty$ along the real axis to be an integral from $-i\xi-\infty$ to $-i\xi+\infty$ will not change the value of the integral, since there are no residues to pick up, while it will convert the integrand back to e^{-x^2}, which does not involve ξ.

As usual, an integral from $-\infty$ to $+\infty$ is a limit of the corresponding integral from $-R$ to $+R$, as $R\to+\infty$. Then

$$\int_{-\infty}^\infty e^{-(x+\frac{i\xi}{2})^2}\,dx \;=\; \lim_R \int_{-R}^R e^{-(x+\frac{i\xi}{2})^2}\,dx \;=\; \int_{-i\xi-R}^{-i\xi+R} e^{-x^2}\,dx$$

Let B_R be the rectangle with vertices $\pm R$ and $-i\xi\pm R$, traced counter-clockwise. The integrals over the ends of the box are easily estimated: since $|e^{-(x+iy)^2}| = e^{-\mathrm{Re}\,(x+iy)^2} = e^{-x^2+y^2}$,

$$\Big|\int_R^{-i\xi+R} e^{-x^2}\,dx\Big| \;\le\; \text{length}\cdot(\text{sup on curve}) \;\le\; |\xi|\cdot e^{-R^2}\cdot e^{\xi^2} \;\longrightarrow\; 0 \qquad (\text{as } R\to+\infty)$$

Thus,

$$0 \;=\; \lim_{R\to\infty} 0 \;=\; \lim_R \int_{B_R} e^{-i\xi x}\,e^{-x^2}\,dx \;=\; \lim_R \Big(e^{-\frac{\xi^2}{4}}\int_{-i\xi-R}^{-i\xi+R} e^{-x^2}\,dx - e^{-\frac{\xi^2}{4}}\int_{-R}^{R} e^{-x^2}\,dx\Big)$$

so

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx \;=\; e^{-\frac{\xi^2}{4}} \cdot \int_{-\infty}^{\infty} e^{-x^2}\,dx \;=\; e^{-\frac{\xi^2}{4}} \cdot \sqrt{\pi}$$

This is worth remembering. ///

[03.9] Compute $\int_{-\infty}^{\infty} e^{-i\xi x}\, xe^{-x^2}\,dx$

This is the Fourier transform of $x \to xe^{-x^2}$. We can reduce it to the previous, slightly simpler, computation by an integration by parts:

$$\int_{-\infty}^{\infty} e^{-i\xi x}\, xe^{-x^2}\,dx \;=\; -\tfrac{1}{2}\int_{-\infty}^{\infty} e^{-i\xi x}\, \frac{d}{dx}e^{-x^2}\,dx \;=\; \tfrac{1}{2}\int_{-\infty}^{\infty} \frac{d}{dx}e^{-i\xi x} \cdot e^{-x^2}\,dx$$

Thus,

$$\int_{-\infty}^{\infty} e^{-i\xi x}\, xe^{-x^2}\,dx \;=\; -\tfrac{1}{2}i\xi \int_{-\infty}^{\infty} e^{-i\xi x}\, e^{-x^2}\,dx \;=\; -\tfrac{1}{2}i\xi \cdot e^{-\frac{\xi^2}{4}} \cdot \sqrt{\pi}$$

The integration by parts device is worth remembering. ///

[03.10] For continuous φ on the unit circle $|z| = 1$, define

$$f_\varphi(z) \;=\; \int_0^{2\pi} \frac{\varphi(e^{i\theta})}{e^{i\theta} - z}\,d\theta \qquad\qquad (\text{for } |z| < 1)$$

Show that $f(z)$ is holomorphic. Give an example of φ *not* identically 0 so that f_φ *is* identically 0.

Use Morera's theorem: with γ be a small counter-clockwise triangle around a given z_o in the open unit disk,

$$\int_\gamma f_\varphi(z)\,dz \;=\; \int_\gamma \int_0^{2\pi} \frac{\varphi(e^{i\theta})}{e^{i\theta} - z}\,d\theta\,dz \;=\; \int_0^{2\pi} \varphi(e^{i\theta}) \Big(\int_\gamma \frac{dz}{e^{i\theta} - z} \Big) d\theta \;=\; \int_0^{2\pi} 0\,d\theta \;=\; 0$$

Morera's theorem says that this vanishing implies holomorphy of f_φ. ///

Note that the given integral is not quite a written-out version of Cauchy's kernel, because $d(e^{i\theta}) = i\theta\, e^{i\theta}\,d\theta$, so a factor of $e^{i\theta}$ is missing. Nevertheless, it's *close*. Thus, various heuristics might suggest making $\varphi(e^{i\theta})$ be the boundary value of an *anti-holomorphic* function such as $F(z) = \overline{z}$. Thus, $\varphi(e^{i\theta}) = F(e^{i\theta}) = e^{-i\theta}$. For $|z| < 1$, expanding a geometric series:

$$f_\varphi(z) \;=\; \int_0^{2\pi} \frac{\varphi(e^{i\theta})}{e^{i\theta} - z}\,d\theta \;=\; \int_0^{2\pi} \frac{e^{-i\theta}}{e^{i\theta} - z}\,d\theta \;=\; \int_0^{2\pi} e^{-i\theta} \frac{e^{-i\theta}}{1 - ze^{-i\theta}}\,d\theta \;=\; \sum_{n=0}^{\infty} \int_0^{2\pi} e^{-2i\theta} \big(ze^{-i\theta}\big)^n\,d\theta$$

$$=\; \sum_{n=0}^{\infty} z^n \int_0^{2\pi} e^{-i(2+n)\theta}\,d\theta \;=\; \sum_{n=0}^{\infty} z^n \cdot 0 \;=\; 0$$

Thus, with hindsight, $\varphi(e^{i\theta}) = 1$ would also have given $f_\varphi = 0$. ///

[03.11] Show that a *real-valued* holomorphic function is constant.

Discussion: There are several possible arguments. First, via Cauchy-Riemann equations: for f real-valued on a neighborhood of z_o, taking a derivative along a *real* direction, but also along a *purely imaginary* direction, gives

$$f'(z_o) \;=\; \lim_{\varepsilon \to 0} \frac{f(z_o + \varepsilon) - f(z_o)}{\varepsilon} \;=\; \lim_{\varepsilon \to 0} \frac{f(z_o + i\varepsilon) - f(z_o)}{i\varepsilon} \qquad\qquad (\text{with } \varepsilon \text{ real})$$

The first limit is real, the second imaginary, so the equality implies that they are both 0. Thus, $f' = 0$, and f is constant. ///

Second, we can use the open mapping theorem: the real line contains no (non-empty) open sets of $\mathbb{C}$, so a real-valued holomorphic functions must be constant. ///

Another kind of argument, applicable to *entire* functions with constrained values: for f entire and real-valued, the function $F(z) = e^{if(z)}$ takes values on the unit circle. In particular, F is *bounded* and entire, so *constant*, by Liouville. Then $0 = F'(z) = if'(z)e^{if(z)}$, so $f'(z) = 0$, and f is constant. ///

[03.12] Show that a holomorphic function f with $|f(z)| = 1$, for all z, is constant.

Discussion: The open mapping succeeds: the unit circle contains no (non-empty) open subsets of $\mathbb{C}$, so any such f is constant. ///

[03.13] Show that a holomorphic function on $\mathbb{C}$ taking values in the upper half-plane is constant.

Discussion: Again, the inverse Cayley map $C^{-1}(z) = \frac{z-i}{-iz+1}$ (or similar) maps the upper half-plane to the unit disk. It is a holomorphic map, and compositions of holomorphic are holomorphic (because the same is true of complex-differentiable maps), so $C^{-1} \circ f$ is entire. It is bounded, because it takes values in the unit disk, so by Liouville it is constant. ///

[03.14] Let C be the usual Cantor set

$$C \;=\; \{x \in [0,1] : \text{the ternary expansion of } x \text{ contains only digits 0 and 2, digit 1}\}$$

where terminal repeating 1's $(\ldots 111111 \ldots)$ are converted to $\ldots 2$. Show that there is no non-constant holomorphic function with real part taking values in C.

Discussion: One decisive approach is to invoke the open mapping theorem: images of opens under non-constant holomorphic functions are open. The Cantor set contains no non-empty open subsets. For that matter, we have already observed that any $\mathbb{R}$-valued holomorphic function is constant, for the same reasons. ///

[03.15] Let f be an entire function such that $f(z+1) = f(z)$ and $f(z+i) = f(z)$ for all z. Show that f is constant.

Discussion: First, the given *periodicity relations* imply that all the values of f are determined by its values on $R = \{z = x + iy : 0 \le x \le 1,\ 0 \le y \le 1\}$: given x, y, there are unique integers m, n such that $m \le x < m+1$ and $n \le y < n+1$. By the obvious induction,

$$f(x+iy) \;=\; f((x-m) + i(y-n))$$

while $0 \le x - m < 1$ and $0 \le y - n < 1$. On the compact set $0 \le x \le 1$ and $0 \le y \le 1$, the continuous function f is *bounded*. Thus, f is entire and bounded, so by Liouville, it is constant. ///
