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Complex analysis examples discussion 04

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/~garrett/

[This document is
http://www.math.umn.edu/~garrett/m/complex/examples_2020-21/cx_discussion_04.pdf]

[04.1] Prove that, given holomorphic f, g on a non-empty open set U, and given a *simple* zero z_o of f, for all small-enough complex ε the zero of $f + \varepsilon g$ nearest z_o is also *simple*.

Discussion: This was proven earlier, but/and is very much worth rehearsing. Let $|z - z_o| = r > 0$ be a small circle so that it and its interior are entirely within U. Shrink $r > 0$ if necessary so that f does not vanish on the circle, by the identity principle. Let $m > 0$ be the minimum of the continuous function $|f|$ on the circle, and M the maximum of the continuous function $|g|$ on the circle. Then, on that circle, for $0 < \varepsilon < \frac{m}{M}$,

$$\Big| f - (f + \varepsilon \cdot g) \Big| \;\le\; \varepsilon \cdot |g| \;<\; \frac{m}{M} \cdot M \;=\; m$$

Thus, Rouché's theorem says that the number of zeros of $f + \varepsilon g$ inside the circle is equal to the number of zeros of f there, namely, one. ///

[04.2] For small $w \in \mathbb{C}$, let $f(w)$ be the simple zero of $z^5 - z + w = 0$ near 0. Determine a few terms of the power series expansion of $f(w)$ at $w = 0$.

Discussion: There are at least two superficially different ways to approach this. In both cases, we presume that the simple zero really is a holomorphic function of w. Indeed, since $w = -z^5 + z$ is a holomorphic function of z, and $(-z^5 + z)' = -5z^4 + 1$ is $1 \neq 0$ at $z = 0$, by the holomorphic inverse function theorem, the simple zero really is a holomorphic function of w near 0.

One way is to let $f(w) = c_1 w + c_2 w^2 + \ldots$, substitute $f(w)$ for z in the equation, and get a recursive expression for c_n. The other is to implicitly differentiate. Perhaps we should try both, to compare the workloads.

Substituting the power series for z,

$$\Big(c_1 w + c_2 w^2 + c_3 w^3 + \ldots\Big)^5 - \Big(c_1 w + c_2 w^2 + c_3 w^3 + \ldots\Big) + w \;=\; 0$$

The lowest-order term of that fifth power only appears at degree 5, so $c_1 = 1$, $c_2 = c_3 = c_4 = 0$, by *uniqueness* of power-series expressions for holomorphic functions. Then the implied relation from the equality of w^5 terms is $c_1^5 - c_5 = 0$, so $c_5 = c_1^5 = 1$.

Subsequently, the first place that c_n appears is in the equality for coefficients of w^n, which in the $f(w)^5$ term can only at highest the coefficient c_{n-4}. Thus, there will be a recursion that produces all subsequent coefficients. Note that existence and uniqueness were not in doubt after invocation of the holomorphic inverse function theorem. ///

The other approach, by implicit differentiation, computing power series coefficients by computing successive derivatives, first notes that $f(0) = 0$ satisfies $f(0)^5 - f(0) + 0 = 0$, so we do take $c_0 = 0$. Differentiating with respect to w,

$$0 \;=\; 5f'(w) \cdot f(w)^4 - f'(w) + 1$$

so

$$f'(w) \;=\; \frac{1}{1 - 5 \cdot f(w)^4}$$

and

$$f'(0) \;=\; \frac{1}{1 - 5 \cdot 0} \;=\; 1$$

Differentiating again,

$$f''(w) \;=\; \frac{-5\cdot 4\cdot f'(w)\cdot f(w)^3}{(1-5\cdot f(w)^4)^2}$$

At $w=0$, with $f(0)=0$, this is 0. And so on. Perhaps this approach makes less clear that several power series coefficients are clearly determined (and are 0), than the substitute-and-solve approach, although that is partly due to the sparseness of the polynomial. ///

[04.3] Exhibit a linear fractional transformation mapping $1,2,3$ to z_1,z_2,z_3.

Discussion: Presumably the z_i are distinct, or else this is impossible. We know the qualitative fact that linear fractional transformations are transitive on triples of distinct points on $\mathbb{CP}^1$, and this question is asking for a formula, which will involve variants of what was classically called the *cross ratio*.

An unglamorous but systematic approach is to map one triple of distinct numbers to $0,1,\infty$, and then back from $0,1,\infty$ to the other, or similar. There are various computational approaches to obtaining $\begin{pmatrix} a & b \\ c & d\end{pmatrix}$ mapping given z_1,z_2,z_3 to $0,1,\infty$. One approach is to first map $z_3\to\infty$ and $z_1\to 0$, which is easily done via $\begin{pmatrix} 1 & -z_1 \\ 1 & -z_3\end{pmatrix}$. This sends $z_2\to\dfrac{z_2-z_1}{z_2-z_3}$. To subsequently send the latter to 1 while stabilizing 0 and ∞, multiply by the multiplicative inverse of the latter complex number. Thus,

$$z \;\longrightarrow\; \frac{z-z_1}{z-z_3} \;\longrightarrow\; \frac{z-z_1}{z-z_3}\cdot\frac{z_2-z_3}{z_2-z_1} \;=\; \frac{z_2-z_3}{z_2-z_1}\begin{pmatrix} 1 & -z_1 \\ 1 & -z_3\end{pmatrix}(z) \qquad \text{(sends } z_1,z_2,z_3 \text{ to } 0,1,\infty)$$

The matrix inverse is

$$\begin{pmatrix} 1 & -z_1 \\ 1 & -z_3\end{pmatrix}^{-1} \;=\; \frac{1}{-z_3+z_1}\begin{pmatrix} -z_3 & z_1 \\ -1 & 1\end{pmatrix}(z)$$

Thus,

$$z \;\longrightarrow\; \begin{pmatrix} -z_3 & z_1 \\ -1 & 1\end{pmatrix}\Big(\frac{z_2-z_1}{z_2-z_3}\cdot z\Big) \qquad \text{(sends } 0,1,\infty \text{ to } z_1,z_2,z_3)$$

We can send $1,2,3$ to $0,1,\infty$ by

$$z \;\longrightarrow\; \frac{z-1}{z-3} \;\longrightarrow\; \frac{z-1}{z-3}\cdot\frac{2-3}{2-1} \;=\; -\frac{z-1}{z-3} \;=\; \begin{pmatrix} -1 & 1 \\ 1 & -3\end{pmatrix}(z)$$

Thus, the composition of the maps $1,2,3$ to $0,1,\infty$ and then to z_1,z_2,z_3 is

$$z \;\longrightarrow\; \begin{pmatrix} -z_3 & z_1 \\ -1 & 1\end{pmatrix}\Big(\frac{z_2-z_1}{z_2-z_3}\cdot\Big(-\frac{z-1}{z-3}\Big)\Big) \qquad \text{(sends } 1,2,3 \text{ to } z_1,z_2,z_3)$$

So-called simplification is most likely misguided. ///

[04.4] Exhibit a linear fractional transformation mapping the circle $|z|=1$ (minus a point) to the line $\mathrm{Re}(z)=\mathrm{Im}(z)$.

Discussion: Use the fact that linear fractional transformations preserve the collection of lines-and-circles, and that a line-or-circle is determined by three points on it, so tracking three points suffices to determine the image. The Cayley map $z\to\dfrac{z+i}{iz+1}$ fixes ± 1, and sends $i\to\infty$, so maps the unit circle to the real line. Then rotate by $e^{i\pi/4}$. Altogether, this is

$$z \;\longrightarrow\; e^{i\pi/4}\cdot\frac{z+i}{iz+1} \;=\; \frac{e^{i\pi/4}z+e^{i\cdot 5\pi/4}}{iz+1} \;=\; \begin{pmatrix} e^{i\pi/4} & e^{i\cdot 5\pi/4} \\ i & 1\end{pmatrix}(z)$$

mapping the unit circle to the diagonal. ///

[04.5] Let $z, z' \in \mathfrak{H}$. Exhibit a linear fractional transformation stabilizing $\mathfrak{H}$ and mapping $z \to z'$.

Discussion: For $z \in \mathfrak{H}$, we have seen that

$$g_z \;=\; \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix} \qquad\qquad (\text{with } z = x + iy \in \mathfrak{H})$$

acting as a linear fractional transformation, maps $i \to z$. So $g_{z'} g_z^{-1}$ maps $z \to i \to z'$. We could multiply out, if we wanted... ///

[04.6] Let z, z' be in the open unit disk $\mathfrak{D}$. Exhibit a linear fractional transformation stabilizing $\mathfrak{D}$ and mapping $z \to z'$.

Discussion: This reduces to the previous example. We have seen that the Cayley map

$$C \;=\; \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$$

acting by linear fractional transformation, gives an isomorphism $\mathfrak{D} \to \mathfrak{H}$. Since $C(0) = i$, with g_w as in the previous example,

$$C^{-1} \circ g_{C(z')} \circ g_{C(z)}^{-1} \circ C$$

sends $z \to z'$. Yes, we could multiply out... ///

[04.7] Exhibit a conformal map of the open half-disk $\{z \in \mathbb{C} : |z| < 1,\ \mathrm{Re}(z) > 0\}$ to the open unit disk.

Discussion: These are two bigons, which we know are conformally equivalent by explicit maps. Namely, first, map the two vertices of the half-disk to 0 and ∞, by $z \to C(z)$, with Cayley map C. Since the images of the two boundary arcs of the half-disk pass through ∞, and linear fractional maps preserve lines-and-circles, the images are (straight) lines passing through 0. Looking at an additional point on both, for example $0 \to C(0) = i$ on the one, and $1 \to C(1) = 1$ on the other, we see that the images are the positive imaginary axis and positive real axis. The squaring map $z \to z^2$ is conformal on the *open* first quadrant, and maps the open first quadrant to $\mathfrak{H}$. Then the inverse Cayley map sends $\mathfrak{H}$ to $\mathfrak{D}$. ///

[04.8] Exhibit a conformal map of the open unit disk, with $[0, 1)$ removed, to the open unit disk.

Discussion: First, rotate $z \to -z$ so that the slit is $(-1, 0]$ instead. Then the principal square root sends $\mathfrak{D} - (-1, 0]$ to the right unit half-disk. Rotate back by $z \to iz$ to obtain $\mathfrak{H}$, and then apply the inverse Cayley map to obtain $\mathfrak{D}$. ///

[04.9] Exhibit a conformal map of the sector $\{re^{i\theta} : r > 0,\ 0 < \theta < \frac{\pi}{4}\}$ to the unit disk.

Discussion: First, the eighth-power map does not quite accomplish this, since the image of this open sector under the eighth power map omits the real interval $[0, 1)$. Instead, the fourth power map $z \to z^4$ does send this sector to the open upper half-plane $\mathfrak{H}$, and then the inverse Cayley map sends $\mathfrak{H}$ to the open unit disk. Thus, $z \to \frac{z^4 - i}{-iz^4 + 1}$ maps the given sector to the open unit disk. ///

[04.10] Exhibit a conformal map from the strip $\{z = x + iy : c < ax + by < c'\}$ to the crescent

$$\Omega \;=\; \{z : |z| < 1,\ |z - \tfrac{1}{2}| > \tfrac{1}{2}\}$$

Discussion: Both regions are examples of *degenerate bi-gons*, namely, where the vertices are *not* distinct points, and, necessarily, the angles at the vertices are 0.

Perhaps it's easier to go in the opposite direction, since it's easier to adjust *strips* by rotations and dilations than to adjust *crescents* by linear fractional transformations stabilizing the outer circle, for example. Thus, map the single vertex $z_1 = 1$ of the crescent to ∞, by $z \to \frac{1}{z-1}$. The image of the outer circle is determined by tracking two more points on it, for example $\pm i$, and the image of the inner by tracking two points on it, for example, 0 and $\frac{1+i}{2}$. That is, the image of the outer circle is the straight line through $\frac{1}{i-1} = \frac{-i-1}{2}$ and $\frac{1}{-i-1} = \frac{i-1}{2}$, while the image of the inner circle is the straight line through $\frac{1}{-1} = -1$ and

$$\frac{1}{\frac{1+i}{2} - 1} \;=\; \frac{2}{1+i-2} \;=\; \frac{2}{i-1} \;=\; -i-1$$

That is, the image of the outer circle is the *vertical* line through $-\frac{1}{2}$, and the image of the inner circle is the vertical line through -1. Thus, the image of the crescent under $z \to \frac{1}{z-1}$ is the strip $\{z : -1 < \mathrm{Re}(z) < -\frac{1}{2}\}$.

Meanwhile, a relation $\{z = x + iy : c < ax + by < c'\}$ with real parameters a, b, c, c' can be rewritten as

$$\{z \;:\; c < \mathrm{Re}\Big(z \cdot (a - ib)\Big) < c'\} \;=\; (a-ib)^{-1} \cdot \{z \;:\; c < \mathrm{Re}(z) < c'\}$$

Further *real* translation and dilation can map any vertical strip to any other:

$$\{z \;:\; c < \mathrm{Re}(z) < c'\} \;=\; c + \{z \;:\; 0 < \mathrm{Re}(z) < c' - c\} \;=\; (c'-c) \cdot \Big(c + \{z \;:\; 0 < \mathrm{Re}(z) < 1\}\Big)$$

In the case at hand, first map by $z \to \frac{1}{z-1}$ to the strip $-1 < \mathrm{Re}(z) < -\frac{1}{2}$, then by $z \to z+1$ to $0 < \mathrm{Re}(z) < \frac{1}{2}$, then by $z \to z/2(c'-c)$ to $0 < \mathrm{Re}(z) < c'-c$, then by $z \to z + c$ to $c < \mathrm{Re}(z) < c'$, then by $z \to (a-bi)^{-1}z$ to $c < ax + by < c'$. ///

[04.11] Let f be holomorphic on $\mathbb{C}$, and meromorphic at infinity, with a pole of order N. Show that f is a polynomial of degree N (and conversely).

Discussion: The converse is relatively easy. Let $f(z) = a_n z^n + \ldots + a_0$, with $a_n \neq 0$, since the degree is n. In the $1/z$ coordinate at ∞,

$$f(1/z) \;=\; a_n(\frac{1}{z})^n + \ldots + a_1 \frac{1}{z} + a_0$$

This is literally a Laurent series expansion with a degree $-N$ term, so is a pole of order/degree n.

Now let f be holomorphic on $\mathbb{C}$, meromorphic at ∞, with a pole of order N. That is, $g(z) = f(1/z)$ has a pole of order N at $z = 0$. That is, there is a constant C such that $|g(z)| \le C \cdot |z|^{-N}$ for $z \to 0$. That is, $|f(z)| \le C \cdot |z|^N$ for $|z| \to +\infty$. Since f is also *entire*, the usual minor extension of Liouville's theorem shows that the derivatives of f at 0 are 0 except for those of order $\le N$, so f is a polynomial of degree N. ///

[04.12] Let holomorphic $f : \mathbb{CP}^1 \to \mathbb{CP}^1$ be 2-to-1. Show that there are two linear fractional transformations α, β such that $\alpha \circ f \circ \beta$ is the map $z \to z^2$.

Discussion: The 2-to-1 property surely counts *multiplicities*.

We have seen that all holomorphic maps $\mathbb{CP}^1 \to \mathbb{CP}^1$ are *rational* maps $f(z) = P(z)/Q(z)$ with polynomials P, Q. Without loss of generality P, Q are relatively prime in the principal ideal domain $\mathbb{C}[X]$. Certainly Q is not identically 0 if the map is not the constant sending all points of $\mathbb{CP}^1$ to ∞. If the degree of P is greater than 2, then (counting multiplicities) more than 2 points map to 0, contradiction. Similarly, if the degree of Q is more than 2, then more than 2 points map to ∞, contradiction.

Let $P(z) = az^2 + bz + c$ and $Q(z) = Az^2 + Bz + C$. Not both a, A can be 0, or else this is a linear fractional transformation, and is not 2-to-1. Post-composing with $z \to 1/z$ if necessary, we can suppose that $A \neq 0$. Then post-compose with a translation to make $a = 0$. This will simplify the algebra. Then

$$(P/Q)'(z) \;=\; \frac{b(Az^2 + Bz + C) - (bz + c)(2Az + B)}{Q^2(z)}$$

The numerator is

$$(bA)z^2 + (bB)z + bC - (2bA)z^2 - (bB + 2cA)z - cB \;=\; (-bA)z^2 + (bB - bB - 2cA)z + (bC - cB)$$

$$=\; (-bA)z^2 + 2(-cA)z + (bC - cB)$$

This has at least one zero unless the coefficients of z^2 and z are both 0, which, since $A \neq 0$, would require that $P(z) = 0$, contradiction.

Thus, there is a zero z_o of the numerator. Then

$$\Big(\frac{P(z)}{Q(z)} - \frac{P(z_o)}{Q(z_o)}\Big)' \;=\; 0$$

so z_o is a *double* zero of $P/Q - P(z_o)/Q(z_o)$, that is, P/Q takes the value $P(z_o)/Q(z_o)$ with multiplicity two at z_o. Pre-composing and post-composing with translations, without loss of generality $z_o = 0$ and $P(z_o)/Q(z_o) = 0$. This reduces to the form $z \to z^2/Q(z)$ with $Q(z) = Az^2 + Bz + C$ with $A \neq 0$ and $C \neq 0$.

Post-composing with $z \to 1/z$, we can consider $f(z) = Q(z)/z^2$, and by post-composing with a translation, $Q(z) = az + b$. If $a = 0$, then $f(z) = b/z^2$, and post-composing with $z \to 1/z$ (and with a dilation) gives $f(z) = z^2$.

With $a \neq 0$, and with $b \neq 0$ to avoid cancellation and reduction to a linear fractional transformation (which would not be 2-to-1), computing a derivative again,

$$f'(z) \;=\; \frac{az^2 - (az+b)2z}{z^4} \;=\; \frac{-az^2 - 2bz}{z^4} \;=\; \frac{-az - 2b}{z^3}$$

This has a zero at $z_o = -2b/a \neq 0$. Thus, $\frac{Q(z)}{z^2} - \frac{Q(z_o)}{z_o^2}$ assumes the value 0 with multiplicity 2 at $z_o \neq 0$. Up to a constant, it is $(z - z_o)^2/z^2 = \Big(\dfrac{z - z_o}{z}\Big)^2$. Pre-composing with the inverse to $z \to (z - z_o)/z$ makes this $f(z) = z^2$. ///
