

(February 2, 2021)

## Complex analysis examples discussion 05

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[This document is http://www.math.umn.edu/~garrett/m/complex/examples\_2020-21/cx\_discussion\_05.pdf]

[05.1] Determine partial fraction expansions for  $\frac{1}{\cos^2 \pi x}$  and  $\tan \pi x$ .

**Discussion:** We could repeat the argument for sine for cosine, *or* we could start with that for sine. We do the latter. Since  $\cos^2 \pi z = \sin^2 \pi(z + \frac{1}{2})$ ,

$$\frac{\pi^2}{\cos^2 \pi z} = \frac{\pi^2}{\sin^2 \pi(z + \frac{1}{2})} = \sum_{n \in \mathbb{Z}} \frac{1}{((z + \frac{1}{2}) - n)^2} = \sum_{n \in \mathbb{Z}} \frac{1}{(z - (n - \frac{1}{2}))^2}$$

Next, we want to take an anti-derivative, but to be confident of convergence we group the  $n - \frac{1}{2}$  and  $-n + \frac{1}{2}$  terms together before taking the antiderivative:

$$\pi \cdot \tan \pi z = - \sum_{n=1,2,\dots} \left( \frac{1}{z - (n - \frac{1}{2})} + \frac{1}{z + (n - \frac{1}{2})} \right)$$

as desired. ///

[05.2] Determine the product expansion for  $\cos \pi x$ .

**Discussion:** In parallel to the derivation for  $\sin^2 \pi z$ , we observe that

$$\frac{d}{dz} \log \cos \pi z = \pi \frac{1}{\cos \pi z} \cdot (-\sin \pi z)$$

And

$$\frac{d}{dz} \log \left( 1 - \frac{z}{n - \frac{1}{2}} \right) = \frac{-1/(n - \frac{1}{2})}{1 - \frac{z}{n - \frac{1}{2}}} = \frac{1}{z - (n - \frac{1}{2})}$$

Anti-differentiating, with some constant  $C$ ,

$$\pi \log \cos \pi z = C + \sum_{n=1,2,\dots} \left( \log \left( 1 - \frac{z}{n - \frac{1}{2}} \right) + \log \left( 1 + \frac{z}{n - \frac{1}{2}} \right) \right)$$

Exponentiating,

$$\pi \cos \pi z = e^C \cdot \prod_{n=1,2,3,\dots} \left( \left( 1 - \frac{z}{n - \frac{1}{2}} \right) \cdot \left( 1 + \frac{z}{n - \frac{1}{2}} \right) \right) = e^C \cdot \prod_{n=1,2,3,\dots} \left( 1 - \frac{z^2}{(n - \frac{1}{2})^2} \right)$$

Looking at the power series coefficient at  $z = 0$ ,  $e^C = \pi$ , so

$$\cos \pi z = \prod_{n=1,2,3,\dots} \left( 1 - \frac{z^2}{(n - \frac{1}{2})^2} \right)$$

is the desired product expansion. ///

[05.3] Exhibit a meromorphic function on  $\mathbb{C}$  with simple poles at points  $\log n$  for  $n = 1, 2, 3, 4, 5, \dots$  and no other poles. Also, *contemplate* the analogous, considerably more difficult, question when the residue is required to be 1 at every pole.

**Discussion:** From the general idea of partial fraction expansions, we'd want to sum terms of the form  $c_n/(z - \log n)$ , with non-zero constants  $c_n$  decreasing rapidly enough to give (absolute, uniform-on-compacts) convergence. For example,

$$\sum_{n=1,2,3,\dots} \frac{1/n^2}{z - \log n}$$

succeeds.

Yes, there are some elementary estimates to do. Although the necessary convergence estimates are not difficult, it is useful to see at least one way to do what needs to be done, rather than merely dismissing it (which doesn't explain anything, and does not give a role model).

Let  $C$  be a compact subset  $C \subset \mathbb{C}$  not containing any  $\log n$ . Let  $R > 0$  be large enough so that  $C \subset \{|z| \leq R\}$ . For  $\log n > 2R$ , by the triangle inequality

$$|z - \log n| \geq |\log n| - |z| \geq 2R - R = R$$

Thus,

$$\left| \sum_{\log n \geq 2R} \frac{1/n^2}{z - \log n} \right| \leq \left| \sum_{\log n \geq 2R} \frac{1/n^2}{R} \right| = \frac{1}{R} \sum_{\log n \geq 2R} \frac{1}{n^2}$$

showing uniform (absolute) convergence for  $z \in C$ .

At the same time, there are only finitely-many  $n$  with  $\log n \leq 2R$ , so the *finite* sum

$$\sum_{\log n < 2R} \frac{1/n^2}{z - \log n}$$

is a *finite* sum of continuous functions in  $z$ , on the compact set  $C$ . Thus, it is continuous (and certainly converges absolutely uniformly for  $z \in C$ ).

Thus, we have a sum that is convergent (absolutely) uniformly on compacts. Thus, by a corollary to Cauchy theorems, it converges to a holomorphic function. ///

[05.4] Check that the Euclidean Laplacian  $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$  on  $\mathbb{R}^n$  is *rotation-invariant*, in the following sense. A *rotation* is a linear map  $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$  preserving the usual inner product  $\langle x, y \rangle = \sum_i x_i y_i$ , and preserving orientations (so  $\det g = 1$ , rather than  $-1$ ). The asserted rotation-invariance is

$$\Delta(f \circ g) = (\Delta f) \circ g \quad (\text{for twice-differentiable } f \text{ and rotation } g)$$

(In fact,  $\Delta$  is also preserved by *reflections*, which are orientation-reversing, so the determinant condition can be safely ignored.)

**Discussion:** The  $n$ -by- $n$  real matrix  $g$  is a rotation-or-reflection when  $g^\top g = 1_n$ . Element-wise, this is

$$\delta_{ik} = \sum_j (g^\top)_{ij} g_{jk} = \sum_j g_{ji} g_{jk} \quad (\text{with Kronecker delta } \delta_{ik})$$

Elements  $x \in \mathbb{R}^n$  can be expressed as either row vectors or column vectors, with  $g$  acting either by right multiplication or left, respectively, without affecting the conclusion. We choose *row* vectors and *right* multiplication:

$$\Delta(f \circ g)(x) = \sum_\ell \left( \frac{\partial}{\partial x_\ell} \right)^2 f(\dots, \sum_i x_i g_{ij}, \dots) = \sum_\ell \frac{\partial}{\partial x_\ell} \sum_s g_{\ell s} f_s(\dots, \sum_j x_i g_{ij}, \dots)$$

where  $f_s$  is the partial derivative of  $f$  with respect to its  $s^{\text{th}}$  argument. Taking the next derivative gives

$$\sum_{\ell} \sum_{s,t} g_{\ell s} g_{\ell t} f_{st}(\dots, \sum_i x_i g_{ij}, \dots)$$

Interchange the order of the sums and use  $\sum_{\ell} g_{\ell s} g_{\ell t} = \delta_{st}$ :

$$\sum_s f_{ss}(\dots, \sum_i x_i g_{ij}, \dots) = (\Delta f)(xg) = ((\Delta f) \circ g)(x)$$

as desired. ///

[05.5] Check that for harmonic  $h$  and holomorphic  $f$ , the composition  $h \circ f$  is invariably harmonic, while  $f \circ h$  need not be. (Yes, much of the issue is suitable formulation of the computation.)

**Discussion:** First the easy part: with holomorphic  $f(z) = z^2$  and harmonic  $h(x + iy) = y$ ,

$$f(h(x + iy)) = f(y) = y^2$$

and  $\Delta y^2 = 2 \neq 0$ , so  $f \circ h$  is not harmonic.

For the more substantive assertion, first, we give an economical complex-function-theory-oriented argument, but which requires some confidence in using  $z$  and  $\bar{z}$ . Then we give a more tangible, but longer, real-variables argument.

As often done, let

$$\partial = \partial/\partial z = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \quad \bar{\partial} = \partial/\partial \bar{z} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)$$

Then  $\Delta = 4\partial\bar{\partial} = 4\bar{\partial}\partial$ . By a form of the chain rule,

$$\frac{1}{4} \Delta(h \circ f) = \partial\bar{\partial}(h(f, \bar{f})) = \partial(\bar{\partial}f \cdot h_1 + \bar{\partial}\bar{f} \cdot h_2) = \partial(0 + \bar{\partial}\bar{f} \cdot h_2) = \partial(\bar{\partial}\bar{f} \cdot h_2)$$

where  $h_1, h_2$  are the partial derivatives of  $h$  with respect to its first and second arguments. The vanishing  $\bar{\partial}f = 0$  follows from the Cauchy-Riemann equation consequence of *holomorphy*. Continuing, using the product rule and chain rule again, this is

$$\partial\bar{\partial}\bar{f} \cdot h_2 + \bar{\partial}\bar{f} \cdot \partial f \cdot h_{21} + \bar{\partial}\bar{f} \cdot \partial\bar{f} \cdot h_{22} = 0 + 0 + 0$$

since  $\partial(\bar{\partial}\bar{f}) = 0$  by anti-holomorphy of  $\bar{\partial}\bar{f}$ , and  $h_{21} = 0$  by harmonic-ness of  $h$ , and  $\partial\bar{f} = 0$  by anti-holomorphy of  $\bar{f}$ . ///

For a real-variables proof that  $h \circ f$  is harmonic, write  $h$  as a function of the real and imaginary parts of a complex number, and write  $f(x + iy) = u(x, y) + iv(x, y)$ . Then

$$\begin{aligned} \Delta(h(u(x, y), v(x, y))) &= \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) h(u(x, y), v(x, y)) \\ &= \frac{\partial}{\partial x} \left( h_1(u, v)u_x + h_2(u, v)v_x \right) + \frac{\partial}{\partial y} \left( h_1(u, v)u_y + h_2(u, v)v_y \right) \\ &= \left( h_{11}u_x^2 + (h_{12} + h_{21})u_x v_x + h_1 u_{xx} + h_{22}v_x^2 + h_2 v_{xx} \right) + \left( h_{11}u_y^2 + (h_{12} + h_{21})u_y v_y + h_1 u_{yy} + h_{22}v_y^2 + h_2 v_{yy} \right) \\ &= h_{11}(u_x^2 + u_y^2) + (h_{12} + h_{21})(u_x v_x + u_y v_y) + h_1(u_{xx} + u_{yy}) + h_{22}(v_x^2 + v_y^2) + h_2(v_{xx} + v_{yy}) \end{aligned}$$

The real and imaginary parts  $u, v$  of  $f$  are themselves harmonic, so the  $h_1$  and  $h_2$  terms vanish, leaving

$$h_{11}(u_x^2 + u_y^2) + (h_{12} + h_{21})(u_x v_x + u_y v_y) + h_{22}(v_x^2 + v_y^2)$$

In terms of real and imaginary parts, the Cauchy-Riemann equation

$$\left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y}\right)(u + iv) = 0$$

becomes

$$u_x + v_y = 0 \quad \text{and} \quad -u_y + v_x = 0$$

Thus, the coefficient of  $h_{12} + h_{21}$  is

$$u_x v_x + u_y v_y = u_x u_y + u_y (-u_x) = 0$$

and

$$h_{11}(u_x^2 + u_y^2) + h_{22}(v_x^2 + v_y^2) = h_{11}(u_x^2 + v_x^2) + h_{22}(v_x^2 + (-u_x)^2) = (h_{11} + h_{22}) \cdot (u_x^2 + v_x^2) = 0 \cdot (u_x^2 + v_x^2)$$

so  $h \circ f$  is holomorphic. ///

[05.6] Show that every harmonic function  $u$  on an annulus  $r < |z| < R$  is of the form

$$u(z) = a_0 + b_0 \log |z| + \sum_{0 \neq n \in \mathbb{Z}} (a_n z^n + b_n \bar{z}^n)$$

for constants  $a_i, b_i$ .

**Discussion:** Use polar coordinates  $z = re^{i\theta}$  on  $0 < |z| < 1$ , and express  $u$  as a Fourier series in  $\theta$  with coefficients that are functions of  $r$ :

$$u(re^{i\theta}) = \sum_{n \in \mathbb{Z}} c_n(r) e^{in\theta}$$

In polar coordinates, with  $u(re^{i\theta}) = f(r, \theta)$ , with  $r = \sqrt{x^2 + y^2}$  and  $\theta = \arctan \frac{y}{x}$ ,

$$r_x = \frac{x}{r} \quad r_y = \frac{y}{r} \quad \theta_x = \frac{-\frac{y}{x^2}}{1 + (\frac{y}{x})^2} = \frac{-y}{r^2} \quad \theta_y = \frac{\frac{1}{x}}{1 + (\frac{y}{x})^2} = \frac{x}{r^2}$$

and the Laplacian is

$$\begin{aligned} \Delta u &= \frac{\partial}{\partial x} (f_r r_x - f_\theta \theta_x) + \frac{\partial}{\partial y} (f_r r_y + f_\theta \theta_y) = \frac{\partial}{\partial x} \left( f_r \frac{x}{r} - f_\theta \frac{y}{r^2} \right) + \frac{\partial}{\partial y} \left( f_r \frac{y}{r} + f_\theta \frac{x}{r^2} \right) \\ &= \left( f_{rr} \left( \frac{x}{r} \right)^2 + f_r \left( \frac{1}{r} - \frac{x^2}{r^3} \right) - (f_{r\theta} + f_{\theta r}) \frac{xy}{r^3} + f_{\theta\theta} \left( \frac{y}{r^2} \right)^2 - f_\theta \frac{2xy}{r^4} \right) \\ &\quad + \left( f_{rr} \left( \frac{y}{r} \right)^2 + f_r \left( \frac{1}{r} - \frac{y^2}{r^3} \right) + (f_{r\theta} + f_{\theta r}) \frac{xy}{r^3} + f_{\theta\theta} \left( \frac{x}{r^2} \right)^2 + f_\theta \frac{2xy}{r^4} \right) \\ &= f_{rr} + \frac{f_r}{r} + \frac{f_{\theta\theta}}{r^2} = \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) f \end{aligned}$$

Applying this to the Fourier expansion, differentiating termwise,

$$0 = \Delta f(r, \theta) = \sum_n \Delta(c_n(r) e^{in\theta}) = \left( c_n'' + \frac{1}{r} c_n' + \frac{1}{r^2} c_n (in)^2 \right) e^{in\theta}$$

By *uniqueness* of Fourier expansions,

$$c_n'' + \frac{1}{r}c_n' - \frac{n^2}{r^2}c_n = 0$$

This equation is of *Euler type*, with indicial equation

$$\alpha(\alpha - 1) + \alpha - n^2 = 0$$

with solutions  $\alpha = \pm n$ . For  $n = 0$ , the root  $\alpha = 0$  is doubled, and solutions of the differential equation are  $r^0 = 1$  and  $r^0 \cdot \log r = \log r$ . For  $n \neq 0$ , the solutions are  $r^n$  and  $r^{-n}$ .

Translating back to  $z$  and  $\bar{z}$  coordinates, we obtain the indicated expansion. ///

[05.7] (*Euler-type equations of second order*) An ordinary differential equation of the form

$$u'' + \frac{b}{x}u' + \frac{c}{x^2}u = 0$$

with constants  $b, c$  is said to be of *Euler type*. Show that it has solutions  $x^\alpha$  and  $x^\beta$  where  $\alpha, \beta$  are solutions of the **auxiliary equation**

$$\lambda(\lambda - 1) + b\lambda + c = 0$$

Show that  $x^\alpha \log x$  is the second solution if the root of the auxiliary equation is *double*, i.e., if  $\alpha = \beta$ . Use the Mean Value Theorem to genuinely prove that there are no other solutions.

**Discussion:** This discussion is just the discussion of constant-coefficient differential equations in different coordinates.

Let  $T = xd/dx$ . In fact, this is a *dilation*-invariant first-order differential operator on  $\mathbb{R} - \{0\}$ . For  $\alpha \in \mathbb{C}$ , certainly  $(T - \alpha)x^\alpha = 0$ . To prove uniqueness on  $(0, +\infty)$ , suppose  $u$  were another solution. Let  $v = u/x^\alpha$ . Then

$$0 = (T - \alpha)u = (T - \alpha)(x^\alpha v) = \alpha x^\alpha v + x^{\alpha+1}v' - \alpha x^\alpha v = x^{\alpha+1}v'$$

Thus,  $v' = 0$  (on any connected interval inside  $(0, +\infty)$ ). By the Mean Value Theorem,  $v$  is constant. This proves uniqueness for the first-order homogeneous equation.

Next, for exponent  $\beta$ ,  $(T - \alpha)x^\beta = (\beta - \alpha) \cdot x^\beta$ . Thus, for  $\alpha \neq \beta$ , the inhomogeneous  $(T - \alpha)u = x^\beta$  has solution  $\frac{x^\beta}{\beta - \alpha}$ . The uniqueness of solution of the *homogeneous* equation  $(T - \alpha)u = 0$  gives most general solution  $\frac{x^\beta}{\beta - \alpha} + C \cdot x^\alpha$  for constant  $C$ , for  $\alpha \neq \beta$ .

To treat the special case  $\alpha = \beta$ , that is,  $(T - \alpha)u = x^\alpha$ , we can *discover* the solution by differentiating  $(T - \alpha)x^\alpha = 0$  with respect to  $\alpha$ :

$$-1 \cdot x^\alpha + (T - \alpha)(\log x \cdot x^\alpha) = 0$$

which gives

$$(T - \alpha)(\log x \cdot x^\alpha) = x^\alpha$$

This trick is part of *variation of parameters*. Even if we see differentiation with respect to  $\alpha$  as only a (very good) heuristic, *after the fact* we can *check* that the latter equation is satisfied.

We can treat higher-order equations  $(T - \alpha)(T - \beta)u = 0$  as iterations of the first-order equations, in the natural fashion: first, this equation is

$$(T - \alpha)\left((T - \beta)u\right) = 0$$

and  $(T - \alpha)v = 0$  has solutions  $C \cdot x^\alpha$  for constants  $C$ . Next, solve

$$(T - \beta)u = x^\alpha$$

For  $\alpha \neq \beta$ , this has solutions  $\frac{x^\alpha}{\beta-\alpha} + C \cdot x^\beta$ , and for  $\alpha = \beta$  has solutions  $\log x \cdot x^\alpha + C \cdot x^\alpha$ . That is, the second-order homogeneous equation has (linearly independent) solutions  $x^\alpha, x^\beta$  for  $\alpha \neq \beta$ , and has solutions  $x^\alpha, \log x \cdot x^\alpha$  for  $\alpha = \beta$ .

These methods apply to higher-order analogous Euler-type equations. ///

[05.8] (*Rotationally invariant harmonic functions in  $\mathbb{R}^n$* ) For  $f$  twice-differentiable on  $\mathbb{R}^n$ , expressible as a (twice-differentiable) function of the *radius*  $r$  alone (at least away from 0), say  $f$  is *spherically symmetric* or *rotationally invariant*. (This could also be formulated as invariance under the action of the orthogonal group by rotations). Show that

$$\Delta f = f'' + \frac{n-1}{r} f'$$

(This is of Euler type). On  $\mathbb{R}^n - \{0\}$ , find two linearly independent harmonic functions.

**Discussion:** Obtaining the expression of the Laplacian for rotationally-invariant functions is a direct computation: letting  $F(x) = f(r)$  with  $r = |x|$ , and  $\partial_i = \partial/\partial x_i$ ,

$$\sum_i \partial_i^2 f = \sum_i \partial_i(\partial_i r \cdot f') = \sum_i \left( \partial_i^2 r \cdot f' + (\partial_i r)^2 \cdot f'' \right)$$

Of course, since  $r^2 = \sum_i x_i^2$ , we have  $r \partial_i r = x_i$  and  $\partial_i r = \frac{x_i}{r}$ , so the coefficient of  $f'$  is

$$\sum_i \partial_i^2 r = \sum_i \partial_i \left( \frac{x_i}{r} \right) = \sum_i \left( \frac{1}{r} - \frac{x_i}{r^2} \cdot \frac{x_i}{r} \right) = \frac{n}{r} - \frac{r^2}{r^3} = \frac{n-1}{r}$$

The coefficient of  $f''$  is

$$\sum_i (\partial_i r)^2 = \sum_i \left( \frac{x_i}{r} \right)^2 = \frac{r^2}{r^2} = 1$$

This gives the asserted formula for the Laplacian on rotationally invariant functions in  $\mathbb{R}^n$ .

The solutions of that Euler-type equation are  $r^0 = 1$  and  $r^{2-n}$ , for  $n \neq 2$ , and  $r^0 = 1$  and  $\log r$  for  $n = 2$ , because the indicial equation has a double root for  $n = 2$ . ///

[05.9] Prove that for non-vanishing entire  $f$ , the function  $F(z) = \int_0^z \frac{f'(w)}{f(w)} dw$  is *entire*, and essentially gives a logarithm of  $f$ , in the sense that  $f(z) = e^{C+F(z)}$  for suitable constant  $C$

**Discussion:** The idea is to observe that  $F' = f'/f$ , from the definition of  $F$ , and take an antiderivative of these holomorphic functions. For some constant  $C$ , *at least locally*,

$$F = C + \log f$$

The constant  $C$  is (locally) certainly ambiguous by integer multiples of  $2\pi i$ . Exponentiating, at least locally,

$$e^F = e^C \cdot e^{\log f} = e^C \cdot f$$

At this point, there is no ambiguity in the values of  $e^F$  nor of  $f$  itself. To determine the constant  $e^C$ , take  $z = 0$  in the defining integral for  $F$ , so

$$F(0) = \int_0^0 \frac{f'}{f} = 0$$

Thus, evaluating at 0,

$$e^0 = e^C \cdot f(0)$$

and  $e^C = f(0)^{-1}$ . Thus, with  $F$  adjusted by a constant,  $e^F = f$ . ///

[05.10] Define  $f$  on the unit circle by  $f(e^{i\theta}) = \theta^2$ , for  $-\pi < \theta < \pi$ . Find a harmonic function  $u$  on the open disk whose boundary values are  $f$ .

**Discussion:** The Fourier coefficients of the function  $x^2$  on  $[-\pi, \pi]$  are (constant multiples of)  $\frac{2\pi}{n^2}$ , for  $n \neq 0$ , and (the same constant multiple of)  $\pi^3/3$  for  $n = 0$ . Thus, the boundary function is (up to a constant)

$$f = \frac{\pi^2}{3} + 2\pi \sum_{n \neq 0} \frac{e^{in\theta}}{n^2}$$

Extrapolating  $e^{in\theta}$  to  $z^n$  for  $n \geq 1$  and to  $\bar{z}^{|n|}$  for  $n < 0$ , we have a harmonic function

$$u(r, \theta) = \frac{\pi^2}{3} + 2\pi \sum_{n \geq 1} \frac{z^n}{n^2} + 2\pi \sum_{n \geq 1} \frac{\bar{z}^n}{n^2}$$

This converges well for  $|z| < 1$ , and, even if we view this argument only as a heuristic, *after the fact* the boundary values can be verified. Indeed, Abel's theorem about non-tangential boundary limits applies to both the  $z$ -part and the  $\bar{z}$ -part. ///

[05.11] The Fourier expansion

$$\delta(\theta) = \sum_{n \in \mathbb{Z}} e^{in\theta} = \sum_{n \in \mathbb{Z}} \widehat{\delta}(n) e^{in\theta} \quad (\text{with } \widehat{\delta}(n) = 1 \text{ for all } n \in \mathbb{Z})$$

certainly does not converge *pointwise*, but does make sense as the expansion of the periodic Dirac  $\delta$ , sometimes called *Dirac comb* function on  $\mathbb{R}/2\pi\mathbb{Z}$ , in the following sense. The *Plancherel identity*

$$\langle u, v \rangle = \frac{1}{2\pi} \int_0^{2\pi} u(\theta) \overline{v(\theta)} d\theta = \sum_{n \in \mathbb{Z}} \widehat{u}(n) \cdot \overline{\widehat{v}(n)} \quad (\text{for } u, v \in L^2(S^1))$$

$L^2(S^1) \times L^2(S^1) \rightarrow \mathbb{C}$  can be restricted in the first argument and extended in the second, so that for *smooth*  $u(\theta) = \sum_{n \in \mathbb{Z}} \widehat{u}(n) e^{in\theta}$ , pairing against  $\delta$  correctly evaluates  $u$  at  $\theta = 0$ :

$$u(0) = \sum_n \widehat{u}(n) e^{in \cdot 0} = \sum_{n \in \mathbb{Z}} \widehat{u}(n) \cdot 1 = \sum_{n \in \mathbb{Z}} \widehat{u}(n) \cdot \overline{\widehat{\delta}(n)} = \langle u, \delta \rangle$$

Identifying the circle with the boundary  $\{z : |z| = 1\}$  of the disk  $\{z : |z| < 1\}$ , determine the harmonic function on the disk whose boundary value function is the periodic Dirac  $\delta$ .

**Discussion:** The boundary value  $f = \sum_{n \in \mathbb{Z}} 1 \cdot e^{in\theta}$  is not convergent pointwise, but *is* convergent as a generalized function (distribution). Also, there is a question of normalizing constant. Not worrying about those details, knowing that  $e^{in\theta}$  is the boundary value of  $e^{in\theta} r^{|n|}$ , let

$$u(r, \theta) = \sum_n 1 \cdot e^{in\theta} r^{|n|}$$

In whatever limit is legitimate as  $r \rightarrow 1^-$ , this approaches  $f = \sum_{n \in \mathbb{Z}} 1 \cdot e^{in\theta}$ . The expression for  $u$  admits an interesting simplification:

$$u(r, \theta) = 1 + \sum_{n \geq 1} z^n + \sum_{n \geq 1} \bar{z}^n = 1 + \frac{z}{1-z} + \frac{\bar{z}}{1-\bar{z}} = \frac{(1-z)(1-\bar{z}) + z(1-\bar{z}) + \bar{z}(1-z)}{(1-z)(1-\bar{z})} = \frac{1-|z|^2}{(1-z)(1-\bar{z})}$$

Thus, even if we only see this computation as a (very good) heuristic, *after the fact* we can verify that the radial limits of  $u(r, \theta)$  are 0 away from  $z = 1$ , at which it blows up. ///