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Complex analysis examples discussion 06

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[06.1] Show that a harmonic function u on $0 < |z| < 1$ such that

$$\int_{0<x^2+y^2<1} |u(x+iy)|^2 \, dx\, dy \;<\; \infty$$

is of the form $u(x+iy) = v(x+iy) + c\log|z|$ for v harmonic on the disk $|z| < 1$, for some constant c.

Discussion: There is a Fourier expansion

$$u(z) \;=\; \sum_{n\in\mathbb{Z}} c_n(r)\, e^{in\theta}$$

and the integral of $|u|^2$ over the punctured disk is

$$\int_{0<r<1} |u|^2 \;=\; \sum_{m,n} \int_0^1 c_m(r)\,\bar{c}_n(r)\Big(\int_0^{2\pi} e^{im\theta}e^{-in\theta}\, d\theta\Big) r\, dr \;=\; 2\pi \sum_n \int_0^1 |c_n(r)|^2 r\, dr$$

by orthogonality of distinct exponentials. From the previous example, the 0^{th} Fourier coefficient is a linear combination $a_0 + b_0 \log r$, and

$$\int_0^1 \Big|a_0 + b_0\log r\Big|^2 \, r\, dr \;<\; \infty \qquad\qquad \text{(for arbitrary } a_0, b_0)$$

In contrast, for $n > 0$,

$$\int_0^1 \Big|a_n r^n + b_n r^{-n}\Big|^2 \, r\, dr \;=\; \int_0^1 \Big(|a_n|^2 r^{2n} + (a_n\bar{b}_n + \bar{a}_n b_n) + |b_n|^2 r^{-2n}\Big)\, r\, dr$$

The first two summands have finite integrals, but $\int_0^1 r^{-2n}\, r\, dr = +\infty$. Thus, $b_n = 0$. That is, apart from the $\log r$ term, the only non-zero coefficients in the Fourier expansion give terms $z^n = r^n e^{-n\theta}$ and $\overline{z}^n = r^n e^{-in\theta}$ with $n \ge 0$. A sum of such terms is harmonic on the whole disk. ///

[06.2] Without citing Weierstraß's or Hadamard's product theorems, show that the infinite-product part $\prod_{n=1}^{\infty}\Big(1+\dfrac{z}{n}\Big)e^{-z/n}$ of $1/\Gamma(z)$ is convergent for all $z \in \mathbb{C}$.

Discussion: It is best to prove that the product is *uniformly* convergent for z in compact subsets of $\mathbb{C}$, so that the infinite product is proven to be a holomorphic functions, *and* with a meaning of *convergent* that does not collapse when one of the factors is 0 for some particular values of the argument z.

For the latter, it is sensible to say that an infinite product $\prod_{n=1}^{\infty}(1 + c_n(z))$ is *absolutely convergent*, and *uniformly* for z in compacts C, when, for every compact C, there are only finitely-many factors $1 + c_n(z)$ that vanish anywhere on C. Second, for every N, large enough so that $1 + c_n(z) \neq 0$ for $n \ge N$ for all $z \in C$, require that $\sum_{n\ge N} |c_n(z)| < +\infty$ and is *uniformly bounded* for $z \in C$.

Given compact $C \subset \mathbb{C}$, let N be large enough so that C is contained in the N-ball centered at 0. It suffices to apply the criterion to the *inverses* $e^{z/n}/(1 + \frac{z}{n})$ of the factors. We have

$$\frac{e^{z/n}}{1+\frac{z}{n}} \;=\; \frac{1 + \frac{z}{n} + \frac{1}{2!}(\frac{z}{n})^2 + \ldots}{1+\frac{z}{n}} \;=\; 1 + \frac{\frac{1}{2!}(\frac{z}{n})^2 + \ldots}{1+\frac{z}{n}}$$

Estimating the sum over $n > N^3$ by an integral

$$\sum n = N^3 + 1^\infty \frac{1}{n^k} \le \int_{N^3} \frac{dt}{t^k} = \frac{1}{(k-1)(N^2)^{k-1}}$$

for $N \ge 2$,

$$\sum_{n>N^3} \Big|\frac{\frac{1}{2!}(\frac{z}{n})^2 + \ldots}{1+\frac{z}{n}}\Big| \le \frac{1}{1-\frac{N}{N^3}}\Big|\frac{1}{2!}(\frac{z^2}{N^3}) + \frac{1}{3!}\frac{z^3}{2(N^3)^2} + \ldots\Big| \le \frac{1}{1-\frac{N}{N^3}}\Big(\frac{1}{2!}(\frac{N^2}{N^3}) + \frac{1}{3!}\frac{N^3}{2(N^3)^2} + \ldots\Big)$$

$$\le 2\cdot\tfrac{1}{2}\cdot\frac{1/N}{1-\frac{1}{N}} \le \frac{2}{N}$$

by estimating by a geometric series:

$$\frac{N^2}{N^3}) + \frac{1}{3!}\frac{N^3}{2(N^3)^2} + \ldots \le \frac{1}{N} + \frac{1}{N^2} + \frac{1}{N^3} + \ldots = \frac{1/N}{1-\frac{1}{N}}$$

That is, letting $1 + c_n(z) = e^{z/n}/(1+\frac{z}{n})$,

$$\sum_{n\ge N^3} |c_n(z)| \le \frac{2}{N} \qquad (\text{for } |z| \le N)$$

The convergence of this sum gives the desired convergence of the product. ///

[06.3] Prove directly that

$$\lim_{N\to+\infty}\prod_{n=1}^N(1+\frac{1}{n}) = +\infty \qquad \text{and} \qquad \lim_{N\to+\infty}\prod_{n=2}^N(1-\frac{1}{n}) = 0$$

Discussion: These telescope usefully, not to be expected generally, but makes this case thunderously clear:

$$\prod_{n=1}^N(1+\frac{1}{n}) = \prod_{n=1}^N\frac{n+1}{n} = \frac{2}{1}\cdot\frac{3}{2}\cdots\frac{N}{N-1}\cdot\frac{N+1}{N} = N+1$$

The limit is obviously $+\infty$. For the second product,

$$\prod_{n=2}^N(1-\frac{1}{n}) = \prod_{n=2}^N\frac{n-1}{n} = \frac{1}{2}\cdot\frac{2}{3}\cdots\frac{N-2}{N-1}\cdot\frac{N-1}{N} = \frac{1}{N}$$

The limit is obviously 0. ///

[06.4] Following Euler, show that $\sum_{p \text{ prime}} \frac{1}{p}$ diverges, by using the Euler product expansion of $\zeta(s)$ and considering $s \to 1^+$ along the real axis.

Discussion For real $s > 1$, in addition to the Euler product, we also need that $\sum_{\ell\ge 2,\,p}\frac{1}{\ell p^{\ell s}}$ with p summed over primes, *converges*, by estimating primes by natural numbers ≥ 2:

$$\sum_{\ell\ge2,\,p}\frac{1}{\ell p^{\ell s}} \le \sum_{\ell\ge2,\,n\ge2}\frac{1}{\ell n^{\ell s}} \le \sum_{\ell\ge2}\frac{1}{\ell}\int_1^\infty\frac{1}{t^{\ell s}}\,dt = \sum_{\ell\ge2}\frac{1}{\ell}\cdot\frac{1}{\ell s-1} < +\infty$$

uniformly in $s > 1$. Letting C be that finite bound,

$$C + \sum_p \frac{1}{p^s} \;\geq\; \sum_p \Big(\frac{1}{p^s} + \frac{1}{2p^{2s}} + \frac{1}{3p^{3s}} + \ldots\Big) \;=\; \sum_p -\log(1-\frac{1}{p^s}) \;=\; \sum_p \log\frac{1}{1-\frac{1}{p^s}} \;=\; \log\prod_p \frac{1}{1-\frac{1}{p^s}}$$

$$= \;\log\zeta(s) \;\geq\; \log\Big(\sum_n \frac{1}{n^s}\Big) \;\geq\; \log\Big(\int_1^\infty \frac{1}{t^s}\,dt\Big) \;=\; \log(\frac{1}{s-1})$$

As $s \to 1^+$, the logarithm blows up, so the sum $\sum_p 1/p$ must be infinite. ///

[06.5] From the fact that the Fourier transform of $1/(1+x^2)$ is essentially $e^{-|x|}$ (compute by residues), use the Poisson summation formula to evaluate $\sum_{n=1}^\infty \frac{1}{1+n^2}$ in elementary terms.

Discussion: First, compute the Fourier transform of $f(x) = 1/(1+x^2)$ precisely: depending on the sign of $\xi \in \mathbb{R}$, we use a contour in the upper or lower half-plane: for $\xi \geq 0$,

$$\widehat{f}(\xi) \;=\; \int_{\mathbb{R}} e^{-2\pi i\xi x}\,\frac{1}{1+x^2}\,dx \;=\; -2\pi i \mathrm{Res}_{x=-i} e^{-2\pi i\xi x}\,\frac{1}{1+x^2} \;=\; -2\pi i e^{-2\pi i\xi(-i)}\,\frac{1}{(-i)-i} \;=\; \pi e^{-2\pi\xi}$$

Since f is *even*, its Fourier transform is even, so $\widehat{f}(\xi) = \pi e^{-2\pi|\xi|}$. By Poisson summation,

$$\sum_{n\in\mathbb{Z}} \frac{1}{1+n^2} \;=\; \sum_{n\in\mathbb{Z}} f(n) \;=\; \sum_{n\in\mathbb{Z}} \widehat{f}(n) \;=\; \pi\sum_{n\in\mathbb{Z}} e^{-2\pi|n|} \;=\; \pi\cdot\Big(\frac{2}{1-e^{-2\pi}} - 1\Big)$$

by summing two geometric series. The precise expression to be evaluated is

$$\sum_{n=1}^\infty \frac{1}{1+n^2} \;=\; -1 + \tfrac{1}{2}\sum_{n\in\mathbb{Z}} \frac{1}{1+n^2} \;=\; -1 + \frac{\pi}{2}\cdot\Big(\frac{2}{1-e^{-2\pi}} - 1\Big)$$

A funny number. ///

[06.6] Check that the Fourier transform of the characteristic function of a symmetric interval $[-a,a]$ is $\dfrac{\sin x}{x}$ (up to constants, which you should determine) (also known as $\mathrm{sinc}\, x$, up to normalizations). Express the *convolution*

$$(u * v)(x) \;=\; \int_{\mathbb{R}} u(x-y)\,v(y)\,dy$$

of two characteristic functions u, v as an explicit piecewise-linear function. Using $\widehat{u*v} = \widehat{u}\cdot\widehat{v}$ (pointwise multiplication), and Poisson summation, express $\sum_n \Big(\dfrac{\sin n}{n}\Big)^2$ in more elementary terms.

Discussion: First, compute carefully the Fourier transform of the characteristic function $\chi_{[-a,a]}$ of the interval $[-a,a]$:

$$\widehat{\chi}(\xi) \;=\; \int_{\mathbb{R}} e^{-2\pi i\xi x}\chi_{[-a,a]}(x)\,dx \;=\; \int_{-a}^a e^{-2\pi i\xi x}\,dx \;=\; \begin{cases} 2a & (\text{for } \xi = 0) \\ \dfrac{e^{-2\pi i\xi a} - e^{-2\pi i\xi(-a)}}{-2\pi i\xi} \;=\; \dfrac{\sin 2\pi a\xi}{\pi\xi} & (\text{for } \xi \neq 0) \end{cases}$$

In fact, $\frac{\sin 2\pi a\xi}{\pi\xi}$ is holomorphic, with a removable singularity at $\xi = 0$. Assuming Poisson summation is applicable,

$$\sum_{n\in\mathbb{Z}} \Big(\frac{\sin 2\pi an}{\pi n}\Big)^2 \;=\; \sum_{n\in\mathbb{Z}} \big(\chi_{[-a,a]}\cdot\chi_{[-a,a]}\big)\widehat{\ }(n) \;=\; \sum_{n\in\mathbb{Z}} \chi_{[-a,a]} * \chi_{[-a,a]}(n)$$

With $a = 1/2\pi$,

$$\sum_{n\in\mathbb{Z}} \Big(\frac{\sin n}{\pi n}\Big)^2 \;=\; \sum_{n\in\mathbb{Z}} \chi_{[-\frac{1}{2\pi},\frac{1}{2\pi}]} * \chi_{[-\frac{1}{2\pi},\frac{1}{2\pi}]}(n)$$

Compute the convolution of $\chi_{[-a,a]}$ with itself:

$$\chi_{[-a,a]} * \chi_{[-a,a]}(x) \;=\; \int_{\mathbb{R}} \chi_{[-a,a]}(x-y)\,\chi_{[-a,a]}(y)\,dy$$

For fixed x, $\chi_{[-a,a]}(x-y)\,\chi_{[-a,a]}(y) = 0$ unless $\max(-a, x-a) \le \min(a, x+a)$, in which case it is 1. Thus,

$$\chi_{[-a,a]} * \chi_{[-a,a]}(x) \;=\; \min(a, x+a) - \max(-a, x-a) \;=\; \begin{cases} 0 & \text{for } x \le -2a \\ 2a+x & \text{for } -2a \le x \le 0 \\ 2a-x & \text{for } 0 \le x \le 2a \\ 0 & \text{for } 2a \le x \end{cases}$$

For $a = 1/2\pi$, this is

$$\chi_{[-\frac{1}{2\pi},\frac{1}{2\pi}]} * \chi_{[-\frac{1}{2\pi},\frac{1}{2\pi}]}(x) \;=\; \begin{cases} 0 & \text{for } x \le -\frac{1}{\pi} \\ \frac{1}{\pi}+x & \text{for } -\frac{1}{\pi} \le x \le 0 \\ \frac{1}{\pi}-x & \text{for } 0 \le x \le \frac{1}{\pi} \\ 0 & \text{for } \frac{1}{\pi} \le x \end{cases}$$

In particular, this convolution is 0 at non-zero integer x. At $x = 0$, it is $1/\pi$. Thus, minding the removable singularity at 0,

$$\sum_{n\in\mathbb{Z}} \Big(\frac{\sin n}{\pi n}\Big)^2 \;=\; \frac{1}{\pi}$$

Multiplying through by π^2, apparently

$$\sum_{n\in\mathbb{Z}} \frac{\sin^2 n}{n^2} \;=\; \pi$$

(assuming applicability of Poisson summation). ///

[06.7] For f a non-vanishing holomorphic function on the open unit disk, show that $z \to \log|f(z)|$ is a well-defined harmonic function.

Discussion: For f non-vanishing, there is a locally well-defined holomorphic $\log f$. The real part of a holomorphic function is harmonic, and the (unambiguous!) real part of $\log\alpha$ is $\log|\alpha|$, with the real-valued logarithm of the positive real $|\alpha|$, so we have harmonic

$$\mathrm{Re}(\log f(z)) \;=\; \log|f(z)|$$

The ambiguities (by $2\pi i\mathbb{Z}$) in $\log f(z)$ are purely imaginary, so taking the real part gives an unambiguous result. ///

[06.8] Prove that $\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s}$ does not vanish in $\mathrm{Re}(s) > 1$.

Discussion: Let $\sigma = \mathrm{Re}(s)$, so $|p^s| = p^\sigma$, and

$$|1 - \frac{1}{p^s}| \;\le\; 1 + |\frac{1}{p^s}| \;=\; 1 + \frac{1}{p^\sigma} \qquad \text{and then} \qquad \frac{1}{|1-\frac{1}{p^s}|} \;\ge\; \frac{1}{1+\frac{1}{p^\sigma}}$$

Taking the logarithm of the Euler product, with $\sigma > 1$, using monotonicity of logarithm,

$$\log|\zeta(s)| \;=\; \log\Big|\prod_p \frac{1}{1-\frac{1}{p^s}}\Big| \;=\; \log\prod_p \frac{1}{|1-\frac{1}{p^s}|} \;=\; \sum_p \log\frac{1}{|1-\frac{1}{p^s}|} \;\ge\; \sum_p \log\frac{1}{1+\frac{1}{p^\sigma}}$$

$$= \sum_p -\log(1+\frac{1}{p^\sigma}) \;=\; \sum_p -\Big(\frac{1}{p^\sigma} - \frac{1}{2p^{2\sigma}} + \frac{1}{3p^{3\sigma}} - \ldots\Big) \;\ge\; -\sum_{\ell\ge 1,\,p} \frac{1}{\ell\,p^{\ell\sigma}} \;\ge\; -\sum_{\ell\ge 1,\,n\ge 2} \frac{1}{\ell\,n^{\ell\sigma}}$$

$$\ge\; -\sum_{\ell\ge 1} \frac{1}{\ell}\int_1^\infty \frac{1}{t^{\ell\sigma}}\,dt \;=\; -\sum_{\ell\ge 1}\frac{1}{\ell}\cdot\frac{1}{\ell\sigma-1}$$

Thus, $\log|\zeta(s)| > -\infty$, so $\zeta(s)\ne 0$ there. ///

[06.9] *(A variant Perron identity)* Show that, for $\sigma > 0$, a vertical path integral moving upward along the line $\mathrm{Re}(s) = \sigma$ evaluates to

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{X^s}{s(s+\theta)}\,ds \;=\; \begin{cases} \frac{1}{\theta}(1-X^{-\theta}) & (\text{for } X>1) \\ 0 & (\text{for } 0<X<1)\end{cases} \qquad (\text{for } \theta>0,\ \sigma>0)$$

The notation is the standard way of indicating a path integral over a vertical line.

Discussion: As with the more delicate Perron identity itself, the *idea* is that the vertical integration contour can be moved to the *right* for $0 < X < 1$, picking up no residues at all, giving 0, and moved to the *left* for $X > 1$, picking up the residues at the simple poles at $s = 0$ and $s = -\theta$. When this is justified, these residues are $X^0/(0+\theta)$ and $X^{-\theta}/(-\theta)$, together giving the indicated $(1-X^{-\theta})/\theta$.

The issue is justification, which is easier in this case than for the original Perron identity, since here we have absolute convergence, unlike the original. That is, first, the indicated integral is absolutely convergent, and can be evaluated as a single limit (rather than worrying about the two tails as separate limits):

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{X^s}{s(s+\theta)}\,ds \;=\; \lim_{T\to+\infty}\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{X^s}{s(s+\theta)}\,ds$$

For $0 < X < 1$, so that X^s decays to the *right*, view the finite vertical integral as the left side of an integral clockwise around the rectangle with vertices $\sigma - iT$, $\sigma + iT$, $T + iT$, $T - iT$, and then back to $\sigma - iT$. Noting that $|X^s| = X^{\mathrm{Re}(s)}$, the integrals over the three other sides are easily estimated: the top and bottom are

$$\Big|\int_{\sigma\pm iT}^{T\pm iT}\frac{X^s\,ds}{s(s+\theta)}\Big| \;\le\; \int_\sigma^T \frac{X^u\,du}{T^2} \;\le\; \int_\sigma^T\frac{1\,du}{T^2} \;\le\; \frac{1}{T} \;\longrightarrow\; 0$$

and the right side is

$$\Big|\int_{T+iT}^{T-iT}\frac{X^s\,ds}{s(s+\theta)}\Big| \;\le\; \int_{-T}^T\frac{X^{-T}\,du}{T^2} \;\le\; \int_{-T}^T\frac{1\,du}{T^2} \;\le\; \frac{2}{T} \;\longrightarrow\; 0$$

Thus, in the case $0 < X < 1$, the vertical integral is the negative (because the path integral is clockwise) sum of residues inside that rectangle, namely, 0.

For $X > 1$, so that X^s decays to the *left*, view the finite vertical integral as the right side of an integral counter-clockwise around the rectangle with vertices $\sigma - iT$, $\sigma + iT$, $-T + iT$, $-T - iT$, and then back to $\sigma - iT$. Much as in the case $0 < X < 1$, the integrals over the three other sides are easily estimated: the top and bottom are

$$\Big|\int_{\sigma\pm iT}^{-T\pm iT}\frac{X^s\,ds}{s(s+\theta)}\Big| \;\le\; \int_{-\sigma}^T\frac{X^{-u}\,du}{T^2} \;\le\; \int_{-\sigma}^T\frac{1\,du}{T^2} \;\le\; \frac{\sigma+T}{T^2} \;\longrightarrow\; 0$$

and the left side is

$$\Big|\int_{-T+iT}^{-T-iT} \frac{X^s\,ds}{s(s+\theta)}\Big| \;\le\; \int_{-T}^{T} \frac{X^{-T}\,du}{(T-\theta)^2} \;\le\; \int_{-T}^{T} \frac{1\,du}{(T-\theta)^2} \;\le\; \frac{2T}{(T-\theta)^2} \;\longrightarrow\; 0$$

Thus, again, the path integral around the rectangle captures the residues inside it, as indicated. ///

[06.10] Let L be an additive subgroup of $\mathbb{R}$, and suppose that L is *discrete* as a subset of $\mathbb{R}$, under the usual topology. Show that either $L = \{0\}$, or $L = \mathbb{Z}\cdot x_o$ for some $x_o \neq 0$.

Discussion: This may be intuitively plausible, but a precise proof is perhaps surprisingly subtle. First, a discrete sub*set* of $\mathbb{R}$ need not be a (topologically) *closed* subset: the set $\{\frac{1}{n} : n = 1,2,3,\ldots\}$ is discrete (every point has a neighborhood which contains no other point of the set), but the set has limit point 0 in $\mathbb{R}$.

But a discrete sub*group* H of $\mathbb{R}$ (or of any *topological group*[1]) *is* discrete, seen as follows. By discreteness, there is a small-enough neighborhood U of $0 \in \mathbb{R}$ such that $U \cap H = \{0\}$. By continuity of the group operation, there is a smaller open neighborhood V of 0 so that $V + V \subset U$. Replacing V by $V \cap (-V)$, we can assume that V is *symmetric*, in the sense that $v \in V$ implies $-v \in V$. Suppose a point $x \in \mathbb{R}$ is in the topological closure of H. If x is not already in H, then for every neighborhood W of 0 there are infinitely-many elements of $H \cap (x + W)$. In particular, there are distinct $h_1, h_2 \in H \cap (x + V)$. That is, there are $v_1, v_2 \in V$ such that $h_i = x + v_i$. Then

$$h_1 - h_2 \;=\; (x+v_1) - (x+v_2) \;=\; v_1 - v_2 \;\in\; V - V \;=\; V + V \;\subset U$$

But $h_1 - h_2 \in H$, and $H \cap U = \{0\}$, so this is impossible. Thus, every limit point of H is already in H. That is, a discrete subgroup H is topologically closed in the ambient topological group.

The set $N = \{h \in H : h > 0\}$ does not meet the neighborhood U of 0, so the closure of N does not contain 0. Since H is not just $\{0\}$ (and since a subgroup of $\mathbb{R}$ is closed under additive inverse), N is non-empty. Thus, its *inf* h_o is in that closure, so is in H, because H is closed. Now we claim that every $h \in H$ is an integer multiple of h_o. It suffices to treat $h > 0$. Let $\ell \cdot h_o$ be the largest integer multiple of h_o less than or equal h, so $\ell h_o \le h < (\ell+1)h_o$. Thus, $0 \le h - \ell h_o < h_o$. Thus $h - \ell h_o$ is a non-negative element of H smaller than h_o, which implies $h - \ell h_o = 0$, since h_o is the smallest strictly positive element of H. ///

[06.11] In the Gaussian integers $\mathbb{Z}[i]$, there are 4 units $\pm 1, \pm i$. The *norm* is $N(m+in) = m^2 + n^2$. Show that the zeta function

$$\zeta_{\mathbb{Z}(i)}(s) \;=\; \frac{1}{\#\mathbb{Z}[i]} \sum_{0\neq m+in\in\mathbb{Z}[i]} \frac{1}{N(m+in)^s} \;=\; \frac{1}{4} \sum_{m,n \text{ not both } 0} \frac{1}{(m^2+n^2)^s}$$

has an analytic continuation and functional equation

$$\pi^{-s}\Gamma(s)\zeta_{\mathbb{Z}[i]}(s) \;=\; \pi^{-(1-s)}\Gamma(1-s)\zeta_{\mathbb{Z}[i]}(1-s)$$

by using

$$\theta(y)^2 \;=\; \Big(\sum_{n\in\mathbb{Z}} e^{-\pi n^2 y}\Big)^2 \;=\; \sum_{m,n\in\mathbb{Z}} e^{-\pi(m^2+n^2)y}$$

[1] A *topological group* G is a (not necessarily abelian) group with a topology so that multiplication and inversion are continuous, as expected. Further, to avoid pathologie while addressing a large family of useful scenarios, G is assumed to be *Hausdorff* and *locally compact*. Thus, oddly, infinite-dimensional topological vectorspaces do not fit this definition. Indeed, topological groups (in this restricted sense) and topological vector spaces have significantly different features.

Discussion: This is completely parallel to Riemann's argument for $\zeta(s)$, using a different but closely related *theta function*:

$$\Theta(y) \;=\; \theta(y)^2 \;=\; \Big(\sum_{m\in\mathbb{Z}} e^{-\pi n^2 y}\Big)^2 \;=\; \sum_{(m,n)\in\mathbb{Z}^2} e^{-\pi(m^2+n^2)y}$$

The theta function $\Theta(y)$ can also be considered directly, using the two-dimensional Fourier transform. The functional equation $\theta(1/y) = \sqrt{y}\cdot\theta(y)$ gives the functional equation

$$\Theta(1/y) \;=\; y\cdot\Theta(y)$$

As in Riemann's discussion, first we have the *integral representation* of $\zeta_{\mathbb{Z}[i]}(s)$ with its appropriate Gamma factor: for $\mathrm{Re}(s) > 1$,

$$\int_0^\infty y^s\,\frac{\Theta(y)-1}{4}\,\frac{dy}{y} \;=\; \frac{1}{4}\sum_{0\neq(m,n)\in\mathbb{Z}^2}\int_0^\infty y^s\,e^{-\pi(m^2+n^2)y}\,\frac{dy}{y}$$

$$=\;\pi^{-s}\cdot\frac{1}{4}\sum_{0\neq(m,n)\in\mathbb{Z}^2}\frac{1}{(m^2+n^2)^s}\int_0^\infty y^s\,e^{-y}\,\frac{dy}{y} \;=\; \pi^{-s}\Gamma(s)\cdot\frac{1}{4}\sum_{0\neq(m,n)\in\mathbb{Z}^2}\frac{1}{(m^2+n^2)^s} \;=\; \pi^{-s}\Gamma(s)\,\zeta_{\mathbb{Z}[i]}(s)$$

To prove the analytic continuation and functional equation, observe that the integral from 1 to ∞

$$\int_1^\infty y^s\,\frac{\Theta(y)-1}{4}\,\frac{dy}{y}$$

extends to an *entire* function, because the rapid decay of $\Theta(y)-1$ dominates the polynomial growth of y^s as $y\to+\infty$.

The integral from 0 to 1 can be converted to an integral from 1 to ∞ via the functional equation of Θ, with some leftover more-elementary terms: replacing y by $1/y$ and then rearranging gives

$$\int_0^1 y^s\,(\Theta(y)-1)\,\frac{dy}{y} \;=\; \int_1^\infty y^{-s}\,(\Theta(1/y)-1)\,\frac{dy}{y} \;=\; \int_1^\infty y^{-s}\,(y\Theta(y)-1)\,\frac{dy}{y}$$

$$=\;\int_1^\infty y^{1-s}\Big((\Theta(y)-1)+(1-\frac{1}{y})\Big)\,\frac{dy}{y} \;=\; \int_1^\infty y^{1-s}\,(\Theta(y)-1)\,\frac{dy}{y}+\int_1^\infty (y^{1-s}-y^{-s})\,\frac{dy}{y}$$

$$=\;\int_1^\infty y^{1-s}\,(\Theta(y)-1)\,\frac{dy}{y}+\frac{1}{s-1}-\frac{1}{s}$$

The integral converges very well for all $s\in\mathbb{C}$, so extends to an entire function, and the two rational functions extend to meromorphic functions on $\mathbb{C}$. Thus,

$$\pi^{-s}\Gamma(s)\,\zeta_{\mathbb{Z}[i]}(s) \;=\; \int_1^\infty (y^s+y^{1-s})\,\frac{\Theta(y)-1}{4}\,\frac{dy}{y}+\frac{1/4}{s-1}-\frac{1/4}{s} \qquad \text{(at first just for } \mathrm{Re}(s)>1\text{)}$$

presents $\pi^{-s}\Gamma(s)\,\zeta_{\mathbb{Z}[i]}(s)$ in a form exhibiting its meromorphic continuation and functional equation $s \longleftrightarrow 1-s$. ///

[06.12] Find a simple trick to express $\wp''$ (for a fixed lattice) as a polynomial in $\wp$.

Discussion: The function $\wp''$ is *even*, so we anticipate it is expressible in terms of $\wp$. Differentiate the Weierstraß relation $\wp'^2 = 4\wp^3-g_2\wp-g_3$, obtaining $2\wp'\wp'' = 12\wp^2\wp'-g_2\wp'$. Cancelling $2\wp'$ gives $\wp'' = 6\wp^2-\frac{g_2}{2}$. ///

[06.13] Fix a lattice L. Express

$$\sum_{\lambda\in L} \frac{1}{(z-\lambda)^4} \qquad\qquad \sum_{\lambda\in L} \frac{1}{(z-\lambda)^6}$$

in terms of $\wp(z)$.

Discussion: These functions are *even*, so they are expressible in terms of $\wp(z)$. One approach, as in the previous example, is to differentiate the partial fraction expansion for $\wp'$, and correcting the constant. From the previous, $\wp'' = 6\wp^2 - \frac{g_2}{2}$, so

$$\sum_{\lambda\in L} \frac{1}{(z-\lambda)^4} \;=\; \frac{1}{3!}\wp'' \;=\; \wp^2 - \frac{g_2}{12}$$

Similarly, differentiating twice more,

$$\sum_{\lambda\in L} \frac{1}{(z-\lambda)^6} \;=\; \frac{1}{5!}\wp''''$$

From $\wp'' = 6\wp^2 - \frac{g_2}{2}$, differentiation twice and replacement of $\wp'^2$ by $4\wp^3 - g_2\wp - g_3$ gives

$$\wp'''' \;=\; (12\wp'\wp)' \;=\; 12(\wp''\wp + \wp'^2) \;=\; 12(\wp''\wp + 4\wp^3 - g_2\wp - g_3)$$

Then

$$\sum_{\lambda\in L} \frac{1}{(z-\lambda)^6} \;=\; \frac{1}{5!}\wp'''' \;=\; \frac{1}{10}(\wp''\wp + 4\wp^3 - g_2\wp - g_3)$$

A more analytic approach would arrange to cancel the poles, leaving an entire, doubly-periodic function, which must be constant, by Liouville. This repeats some of the work in getting to the Weierstraß equation. From the computations in the proof of the Weierstraß equation, the Laurent expansions of f at 0 is

$$\wp(z) \;=\; \frac{1}{z^2} + \frac{g_2}{20}z^2 + O(z^4)$$

Taking derivatives term-wise,

$$f(z) \;=\; \frac{1}{6}\wp'' \;=\; \frac{1}{z^4} + \frac{g_2}{60} + O(z^2)$$

Squaring,

$$\wp(z)^2 \;=\; \frac{1}{z^4} + \frac{g_2}{10} + O(z^2)$$

and

$$f(z) - \wp(z)^2 \;=\; -\frac{g_2}{12} + O(z^2)$$

By Liouville,

$$f(z) - \wp(z)^2 \;=\; -\frac{g_2}{12}$$

which agrees with the previous. The analogous computation for $\wp''''$ is burdensome in comparison to using the Weierstraß equation, which does already package up the relations. ///

[06.14] Fix a lattice Λ. Show that there is *no* elliptic function for Λ with exactly one pole (modulo Λ), and with that pole being *simple*.

Discussion: Let f be an elliptic function for Λ with exactly one pole (mod Λ), at z_o. Let F be a period parallelogram (that is, one choice of fundamental domain) for Λ, with sides (in order) $s_1, s_2, s_1+\lambda_1, s_2+\lambda_2$, for suitable elements λ_1, λ_2 of Λ. Let γ be the boundary of F traced counter-clockwise, tracing s_1, s_2, and then $s_1 + \lambda_1, s_2 + \lambda_2$ *backward*. If γ runs through any $z_o + \Lambda$, indent γ suitably to avoid these points. On one hand, the integral $\int_\gamma f$ is the sum of residues of f inside γ. On the other hand,

$$\int_\gamma f \;=\; \int_{s_1} f + \int_{s_1} f - \int_{s_1+\lambda_1} f - \int_{s_2+\lambda_2} f$$

with the signs due to tracing the translated side $s_i + \lambda_i$ in the opposite direction that s_i is traced. By Λ-periodicity of f, this is

$$\int_{s_1} f(z)\,dz + \int_{s_1} f(z)\,dz - \int_{s_1} f(z+\lambda_1)\,dz - \int_{s_2} f(z+\lambda_2)\,dz$$

$$= \int_{s_1} f(z)\,dz + \int_{s_1} f(z)\,dz - \int_{s_1} f(z)\,dz - \int_{s_2} f(z)\,dz \;=\; 0$$

That is, the integrals over opposite sides of F *cancel*, since the values of f are the same, but the paths are traced in opposite directions. Thus, the residue of f at z_o is 0. This is impossible for a simple pole. ///
