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Complex analysis examples discussion 07

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/˜garrett/

[Draft]

[This document is http://www.math.umn.edu/˜garrett/m/complex/examples_2020-21/cx_discussion_07.pdf]

[07.1] Let f be an entire function such that $|f(z)| \le e^{\mathrm{Re}(z)}$ for all z. Show that $f(z) = c \cdot e^z$ for some constant c with $|c| \le 1$. (The latter special case was on MathStackExchange, math.stackexchange.com/questions/4082085/.) More generally, suppose two entire functions f, g satisfy $|f(z)| \le |g(z)|$ for all z, and show that $f(z) = c \cdot g(z)$ for some constant c with $|c| \le 1$. (Be careful about the zeros of g.)

Discussion: Since $|e^z| = e^{\mathrm{Re}(z)}$, the hypothesis is that $|f(z)| \le |e^z|$. Dividing by e^z (which does not vanish), $|f(z)/e^z| \le 1$. By Liouville, $f(z)/e^z = c$ for some constant, and necessarily $|c| \le 1$. That is, $f(z) = c \cdot e^z$.

The more general case requires a little further attention to detail. Namely, with $|f(z)| \le |g(z)|$, we must first show that the meromorphic function f/g, with isolated singularities at the zeros of g, has only *removable* singularities. Indeed, in a small-enough punctured (to avoid containing any *other* zeros of g) neighborhood P of a zero z_o, $|f(z)/g(z)|$ is *bounded* by 1. This meets one of the criteria for a removable singularity. Further, either by a part of that argument, or from the maximum modulus principle, the extended version of f/g at z_o is still bounded by 1. Thus, the extended version of f/g is entire, and bounded by 1, so by Liouville is constant. ///

[07.2] Show that the group of automorphisms of the field of rational functions $\mathbb{C}(z)$ over $\mathbb{C}$ (that is, bijections $\varphi : \mathbb{C}(z) \to \mathbb{C}(z)$ which preserve addition and multiplication of rational functions, and are the identity map on the subfield $\mathbb{C}$), is the group

$$PGL_2(\mathbb{C}) \;=\; GL_2(\mathbb{C})/Z \;=\; \{\text{multiplicatively invertible complex matrices modulo the center } Z\}$$

(the center Z is the subgroup of scalar matrices) acting by linear fractional transformations $z \to \frac{az+b}{cz+d}$.

Discussion: In fact, the same assertion is true with any field in place of $\mathbb{C}$. That is, neither metric completeness, nor algebraic closedness, nor characteristic 0 is used in the proof, although this issue surely arose historically in a geometric context for $\mathbb{C}(z)$. That is, historically, $\mathbb{C}(z)$ would have been thought of as rational *functions* of z (with coefficients in $\mathbb{C}$), rather than as the field of fractions of the polynomial ring $\mathbb{C}[z]$, thought of as the *free commutative* $\mathbb{C}$*-algebra in one generator*.

A field automorphism of $\mathbb{C}(z)$ over $\mathbb{C}$ is a bijection $\varphi : \mathbb{C}(z) \to \mathbb{C}(z)$ which is the identity map when restricted to $\mathbb{C}$, and preserves addition and multiplication (so, likewise, subtraction and division, since it is a bijection). Every element of $\mathbb{C}(z)$ is (by its description) a rational function $P(z)/Q(z)$ of z. (Here P, Q are polynomials, Q is not identically 0, and P, Q are not both constants.) Thus, an automorphism φ of $\mathbb{C}(z)$ is completely determined by the image of $\varphi(z)$, namely, $\varphi(P(z)/Q(z)) = P(\varphi(z))/Q(\varphi(z))$. What is needed is a condition on φ for bijectivity.

On one hand, every linear fractional transformation $\varphi(z) = g(z) = \frac{az+b}{cz+d}$ gives such a field automorphism of $\mathbb{C}(z)$, by the formula $\varphi(P(z)/Q(z)) = P(\varphi(z))/Q(\varphi(z))$: the linear fractional transformation given by the matrix inverse g^{-1} sends $g(z)$ to z, giving a two-sided inverse to φ, proving bijectivity. The (invertible) scalar matrices act trivially on z, so act trivially as automorphisms of $\mathbb{C}(z)$.

On the other hand, for φ to be a bijection, $\varphi(z)$ must generate the same field extension of $\mathbb{C}$ as does z. So z itself must be a rational function of the (rational function) $\varphi(z)$. In particular, since the function

z is injective, $\varphi(z)$ must be injective. For $\varphi(z) = P(z)/Q(z)$, the injectivity requires (for example) that $P(z)/Q(z) = 0$ has at most a single solution z_o. Thus, $P(z)$ is linear or a constant. Post-composing the given automorphism φ with the automorphism $z \to 1/z$ gives another automorphism $\psi(z) = Q(z)/P(z)$, and, by injectivity considerations, $Q(z)$ is of degree at most 1. Not both P, Q can be constant, by injectivity, so $\varphi(z) = P(z)/Q(z)$ is exactly a linear fractional transformation. ///

[07.3] For a fixed lattice, express $\wp(2z)$ and $\wp(3z)$ as rational functions of $\wp(z)$, using the near-algorithm that is used to prove that the field of elliptic functions for a fixed lattice is $\mathbb{C}(\wp, \wp')$. Contemplate the analogue for $\wp(nz)$.

Discussion: Standard, but not so easy:

Let ω_1, ω_2 be a basis for the lattice Λ. Since $\wp(2z)$ has double poles at $\Lambda/2$, with residues $1/4$, $\wp(2z) - \frac{1}{4}\wp(z)$ has double poles exactly at $\frac{\omega_1}{2} + \Lambda$, $\frac{\omega_2}{2} + \Lambda$, and $\frac{\omega_1+\omega_2}{2} + \Lambda$.

Let a be any one of $\omega_1/2$, $\omega_2/2$, or $(\omega_1 + \omega_2)/2$. Since $\wp(z) - \wp(a)$ is still *even*, and has a zero at $z = a$, this is a *double* zero. Since the number of poles is equal to the number of zeros, and $\wp(z)$ has a double pole, there are no other zeros of $\wp(z) - \wp(a)$. Thus,

$$\big(\wp(2z) - \frac{1}{4}\,\wp(z)\big) \cdot \big((\wp(z) - \wp(\frac{\omega_1}{2}))(\wp(z) - \wp(\frac{\omega_2}{2}))(\wp(z) - \wp(\frac{\omega_1 + \omega_2}{2}))\big)$$

is an even function that has poles only on the lattice.

Again, the Laurent expansion of $\wp(z)$ is the power series of $\wp(z) - 1/z^2$ plus $1/z^2$, and the coefficients of the power series can be computed via derivatives, giving

$$\frac{1}{z^2} + 3E_4 z^2 + 5E_6 z^4 + 7E_8 z^6 + \ldots$$

Thus,

$$\wp(2z) - \frac{1}{4}\,\wp(z) \;=\; 3E_4(2^2 - 1)z^2 + 5E_6(2^3 - 1)z^4 + 7E_8(2^8 - 1)z^6 + \ldots$$

Thus, the (at least) double zero partly cancels the order-six pole, and the order of pole of the adjusted function is at most 4. Via Liouville's theorem, it is inevitably a *polynomial* in $\wp(z)$, of degree at most 2:

$$\big(\wp(2z) - \frac{1}{4}\,\wp(z)\big) \cdot \big((\wp(z) - \wp(\frac{\omega_1}{2}))(\wp(z) - \wp(\frac{\omega_2}{2}))(\wp(z) - \wp(\frac{\omega_1 + \omega_2}{2}))\big) \;=\; A\wp(z)^2 + B\wp(z) + C$$

This leaves 3 constants A, B, C to be determined. The leading $1/z^4$ coefficient of the Laurent expansion is the z^2 coefficient of $\wp(2z) - \wp(z)/4$, namely, $9E_4$. This is the constant A, so

$$\big(\wp(2z) - \frac{1}{4}\,\wp(z)\big) \cdot \big((\wp(z) - \wp(\frac{\omega_1}{2}))(\wp(z) - \wp(\frac{\omega_2}{2}))(\wp(z) - \wp(\frac{\omega_1 + \omega_2}{2}))\big) - 9E_4\wp(z)^2 \;=\; B\wp(z) + C$$

Evaluating the equation at a zero z_o of $\wp(z)$ gives

$$-\wp(2z_o) \cdot \wp(\frac{\omega_1}{2}) \cdot \wp(\frac{\omega_2}{2}) \cdot \wp(\frac{\omega_1 + \omega_2}{2}) \;=\; C$$

If $\wp(2z_o)$ has a pole at $z = z_o$, then z_o is among the 2-division-point values $\omega_1/2, \omega_2/2, (\omega_1 + \omega_2)/2$, and this evaluation requires a little more effort (which we do not exert here).

To determine B, we could take a derivative of the equation and evaluate anywhere $\wp'(z) \neq 0$, ...

If we cared more, we could pursue this or various other possibilities, such as multiplying out the Laurent expansions at 0 and comparing terms. ///

[07.4] Let $v_1, \ldots, v_n \in \mathbb{R}^n$ be linearly independent over $\mathbb{R}$. Show that there is a neighborhood U of $0 \in \mathbb{R}^n$ such that

$$U \cap (\mathbb{Z}v_1 + \ldots + \mathbb{Z}v_n) \;=\; \{0\}$$

Discussion: One approach is to reduce to the case that the v_j are the standard basis e_j of $\mathbb{R}^n$, as follows, by showing the uniform comparability of two metrics on $\mathbb{R}^n$. This comparability result is more broadly useful, as well.

For the standard basis $\{e_1, \ldots, e_n\}$ (where e_j has a 1 at the j^{th} place, and 0s otherwise), the lattice is $\mathbb{Z}^n$, and the proof of the assertion is straightforward, as follows. Let $|(x_1, \ldots, x_n)| = \sqrt{x_1^2 + \ldots + x_n^2}$ be the standard length of vectors, giving the standard metric and topology on $\mathbb{R}^n$. Taking U to be the open unit ball, $U \cap \mathbb{Z}^n = \{0\}$, by the obvious estimate

$$\inf_{0 \neq x \in \mathbb{Z}^n} |x| \;=\; \inf_{0 \neq x \in \mathbb{Z}^n} \sqrt{x_1^2 + \ldots + x_n^2} \;\geq\; \inf_{0 \neq x \in \mathbb{Z}^n} \max_{1 \leq j \leq n} |x_j| \;\geq\; \inf_{0 \neq x \in \mathbb{Z}^n} 1 \;=\; 1$$

Now the reduction to the case $L = \mathbb{Z}^n$. The linear independence is equivalent to the invertibility of the n-by-n matrix g formed with the vectors v_j as columns. The closed unit ball B in $\mathbb{R}^n$ is *compact*, so $\mu = \sup_{|x| \leq 1} |gx|$ and $\nu = \sup_{|x| \leq 1} |g^{-1}x|$ are finite. Thus, for arbitrary $0 \neq x \in \mathbb{R}^n$,

$$|gx| \;=\; \Big| |x| \cdot g \frac{x}{|x|} \Big| \;=\; |x| \cdot \Big| g \frac{x}{|x|} \Big| \;\leq\; \mu \cdot |x|$$

That is, $\mu^{-1} \cdot |gx| \leq |x|$. Similarly, $|g^{-1}x| \leq \nu \cdot |x|$. Replacing x by gx in the latter gives $|x| \leq \nu \cdot |gx|$. Thus, the two norms $x \to |gx|$ and $x \to |x|$ are uniformly comparable:

$$\mu^{-1} \cdot |gx| \;\leq\; |x| \;\leq\; \nu \cdot |gx| \qquad\qquad \text{(for all } x \in \mathbb{R}^n\text{)}$$

(Only one of these inequalities is essential for the present issue.)

With the columns of g being the v_j's, of course $gx = \sum_j x_j v_j$, where the x_j's are the components of x. Thus, we want to know that $r = \inf_{0 \neq x \in \mathbb{Z}^n} |gx|$ is positive, so then the intersection of L with the unit r-ball is just $\{0\}$. Using the uniform comparison of $|gx|$ and $|x|$,

$$\inf_{0 \neq x \in \mathbb{Z}^n} |gx| \;\geq\; \inf_{0 \neq x \in \mathbb{Z}^n} \nu^{-1} \cdot |x| \;=\; \nu^{-1} \cdot \inf_{0 \neq x \in \mathbb{Z}^n} |x| \;=\; \nu^{-1} \cdot 1$$

Thus, the open ν^{-1}-ball meets L just at $\{0\}$. ///

[07.5] Let L be a lattice in $\mathbb{R}^n$, that is, $L = \mathbb{Z}v_1 + \ldots + \mathbb{Z}v_n$, where the v_j are linearly independent over $\mathbb{R}$. Show that

$$\sum_{0 \neq \lambda \in L} \frac{1}{|\lambda|^s} \qquad\qquad \text{(for } s \in \mathbb{C}\text{)}$$

is absolutely convergent for $\mathrm{Re}(s) > n$. Do *not* blithely invoke some apocryphal *integral test in several variables*. Yes, for this example, the natural heuristic gives the truth, and that's a good thing. But we would *like* to prove that that heuristic gives a *proof* of the conclusion.

Discussion: Yes, the heuristic, imagining that $\sum 1/|\lambda|^s$ converges if and only if $\int_{|x| \geq 1} 1/|x|^s \, dx$ converges, regardless of choice of lattice L, *does* gives the correct exponent: converting to spherical coordinates in the integral,

$$\int_{|x| \geq 1} \frac{1}{|x|^s} \, dx \;=\; C \cdot \int_0^\infty \frac{1}{r^s} r^{n-1} \, dr \;=\; C \cdot \int_0^\infty \frac{1}{r^{s-n+1}} \, dr$$

where C is the area of the $n-1$ sphere. This one-dimensional integral is convergent if and only if $\mathrm{Re}(s - n + 1) > 1$, which is $\mathrm{Re}(s) > n$. This heuristic is the way to remember the correct exponent.

Since the absolute value of $|\lambda|^s$ is $|\lambda|^{\mathrm{Re}(s)}$, it suffices to consider $s \in \mathbb{R}$. For $L = \mathbb{Z}^n$, the comparison in one direction is

$$\sum_{0\neq\lambda\in\mathbb{Z}^n} \frac{1}{|\lambda|^s} \;\le\; \sum_{0\neq\lambda\in\mathbb{Z}^n} \frac{1}{|\max_j \lambda_j|^s} \;=\; \sum_{1\le\ell\in\mathbb{Z}} \Big(\frac{1}{\ell^s} \sum_{\lambda:\max_j|\lambda_j|=\ell} 1\Big)$$

The number of $\lambda \in \mathbb{Z}$ with $\max_j |\lambda_j| = \ell$ has an easy upper bound, the product of the number of faces of an n-cube and the number of lattice points on each face. This is $2n \cdot (2\ell)^{n-1}$. Thus,

$$\sum_{0\neq\lambda\in\mathbb{Z}^n} \frac{1}{|\lambda|^s} \;\le\; \sum_{1\le\ell\in\mathbb{Z}} \frac{2n\cdot(2\ell)^{n-1}}{\ell^s} \;=\; 2n\cdot 2^{n-1}\cdot \sum_{1\le\ell\in\mathbb{Z}} \frac{1}{\ell^{s-n+1}}$$

The usual integral test shows that this converges for $s - n + 1 > 1$, which is $s > n$. For the comparison in the other direction,

$$\sum_{0\neq\lambda\in\mathbb{Z}^n} \frac{1}{|\lambda|^s} \;\ge\; \sum_{0\neq\lambda\in\mathbb{Z}^n} \frac{1}{\sqrt{n\cdot\max_j\lambda_j^2}^{\,s}} \;=\; \frac{1}{n^{s/2}} \sum_{0\neq\lambda\in\mathbb{Z}^n} \frac{1}{\sqrt{\max_j\lambda_j^2}^{\,s}} \;\ge\; \frac{1}{n^{s/2}} \sum_{1\le\ell\in\mathbb{Z}} \Big(\frac{1}{\ell^s} \sum_{\lambda:\max_j|\lambda_j|=\ell} 1\Big)$$

The number of $\lambda \in \mathbb{Z}$ with $\max|\lambda_j| = \ell$ has an easy lower bound, the product of the number of faces of an n-cube and the number of lattice points *strictly* on each face, that is, not shared with any other face. This is $2n \cdot (2\ell - 2)^{n-1}$. Thus,

$$\sum_{0\neq\lambda\in\mathbb{Z}^n} \frac{1}{|\lambda|^s} \;\ge\; \frac{1}{n^{s/2}}\cdot 2^{n-1}\cdot \sum_{1\le\ell\in\mathbb{Z}} \frac{(\ell-1)^{n-1}}{\ell^s} \;\ge\; \frac{1}{n^{s/2}}\cdot 2^{n-1}\cdot \sum_{2\le\ell\in\mathbb{Z}} \frac{(\ell/2)^{n-1}}{\ell^s} \;=\; \frac{1}{n^{s/2}}\cdot \sum_{2\le\ell\in\mathbb{Z}} \frac{1}{\ell^{s-n+1}}$$

giving the other implication.

The comparison of norms from an earlier example reduces to the case $L = \mathbb{Z}^n$: for some constants A, B, with real s, with $g\mathbb{Z}^n = L$,

$$\sum_{0\neq\lambda\in L} \frac{1}{|\lambda|^s} \;=\; \sum_{0\neq x\in\mathbb{Z}^n} \frac{1}{|gx|^s} \;\le\; \sum_{0\neq x\in\mathbb{Z}^n} \frac{1}{(A|x|)^s} \;=\; \frac{1}{A^s} \sum_{0\neq x\in\mathbb{Z}^n} \frac{1}{|x|^s}$$

and, in the other direction,

$$\sum_{0\neq\lambda\in L} \frac{1}{|\lambda|^s} \;=\; \sum_{0\neq x\in\mathbb{Z}^n} \frac{1}{|gx|^s} \;\ge\; \sum_{0\neq x\in\mathbb{Z}^n} \frac{1}{(B|x|)^s} \;=\; \frac{1}{B^s} \sum_{0\neq x\in\mathbb{Z}^n} \frac{1}{|x|^s}$$

Thus, (absolute) convergence of the series for $\mathbb{Z}^n$ is equivalent to that for any other lattice. ///

[07.6] From $\wp'' = 6\wp^2 - \frac{g_2}{2}$, looking at the Laurent expansion of both sides at $z = 0$, explicitly determine the rational constant C so that

$$\sum_{0\neq\lambda\in L} \frac{1}{\lambda^8} \;=\; C\cdot\Big(\sum_{0\neq\lambda\in L} \frac{1}{\lambda^4}\Big)^2$$

Discussion: As in the notes, the Laurent expansion of $\wp(z)$ near 0 is the polar term $1/z^2$ plus the power series expansion there of

$$\sum_{0\neq\lambda\in L} \Big(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}\Big)$$

The power series coefficients are computed by Taylor-Maclaurin/Cauchy formulas. The constant coefficient is 0, due to the subtraction. Also, the even-ness of $\wp$ implies the vanishing of all odd-order Laurent coefficients. For $0 < 2n \in 2\mathbb{Z}$, differentiation termwise gives

$$2n^{th} \text{ coefficient} \;=\; \frac{1}{(2n)!}\Big(\frac{d}{dz}\Big)^{2n}\Big|_{z=0} \sum_{0\neq\lambda\in L} \Big(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}\Big)$$

$$= \frac{1}{(2n)!}(2n+1)! \sum_{0 \neq \lambda \in L} \frac{1}{\lambda^{2+2n}} = (2n+1) \sum_{0 \neq \lambda \in L} \frac{1}{\lambda^{2+2n}}$$

Let $\widetilde{E}_{2+2n}$ (depending on L) denote that last sum. Thus, near $z = 0$,

$$\wp(z) = \frac{1}{z^2} + \sum_{2n:2n \geq 2} (2n+1)\widetilde{E}_{2+2n} \cdot z^{2n} = \frac{1}{z^2} + 3\widetilde{E}_4 z^2 + 5\widetilde{E}_6 z^4 + 7\widetilde{E}_8 z^6 + \ldots$$

On one hand, squaring,

$$\wp^2(z) = \frac{1}{z^4} + 6\widetilde{E}_4 + 10\widetilde{E}_6 z^2 + \big((3\widetilde{E}_4)^2 + 14\widetilde{E}_8\big)z^4 + \ldots$$

On the other hand, differentiating twice termwise,

$$\wp''(z) = \frac{6}{z^4} + \sum_{2n \geq 2} (2n+1)2n(2n-1)\widetilde{E}_{2+2n} \cdot z^{2n-2} = \frac{6}{z^4} + 6\widetilde{E}_4 + 60\widetilde{E}_6 z^2 + 210\widetilde{E}_8 z^4 + \ldots$$

Then

$$0 = \wp''(z) - 6\wp^2 + \frac{g_2}{2} = \wp''(z) - 6\wp^2 + \frac{60\widetilde{E}_4}{2}$$

$$= \Big(\frac{6}{z^4} + 6\widetilde{E}_4 + 60\widetilde{E}_6 z^2 + 210\widetilde{E}_8 z^4 + \ldots\Big) - \Big(\frac{6}{z^4} + 36\widetilde{E}_4 + 60\widetilde{E}_6 z^2 + 6\big((3\widetilde{E}_4)^2 + 14\widetilde{E}_8\big)z^4 + \ldots\Big) + \frac{60\widetilde{E}_4}{2}$$

$$= (210\widetilde{E}_8 - 54\widetilde{E}_4^2 - 84\widetilde{E}_8)z^4 + \ldots$$

Thus, the latter coefficient of z^4 is 0:

$$54\widetilde{E}_4^2 = (210 - 84)\widetilde{E}_8 = 126\widetilde{E}_8$$

and, if we've not made an arithmetic error,

$$\widetilde{E}_8 = \frac{54}{126} \cdot \widetilde{E}_4^2 = \frac{3}{7} \cdot \widetilde{E}_4^2$$

Even if there is an arithmetic error, it is clear, qualitatively, that the constant is *rational*. ///

[07.7] From the previous, using Fourier expansions of Eisenstein series

$$\widetilde{E}_{2k}(z) = \sum_{c,d \text{ not both } 0} \frac{1}{(cz+d)^{2k}}$$

determine the rational constant D such that

$$\zeta(8) = D \cdot \zeta(4)^2$$

Thus, from the relatively well-known value $\zeta(4) = \pi^4/90$ (not as well known as $\zeta(2) = \pi^2/6$), we have an expression for the less-well-known $\zeta(8)$ in elementary terms. (Amazing that elliptic functions should entail relations among values of zeta!)

Discussion: *[... iou ...]*

[07.8] For $0 \leq n \in \mathbb{Z}$, let $\psi_n(z) = e^{2\pi i n z}$. In analogy with formation of Eisenstein series, the n^{th} holomorphic *Poincaré series* of weight $2k$ is (in group-theoretic form)

$$P_n(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \frac{1}{(cz+d)^{2k}} \psi_n(\gamma z) \qquad \text{(with } \gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix}\text{)}$$

and $\Gamma_\infty = \{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\} \subset \Gamma$ as usual. Since $\psi_0 = 1$, certainly $P_0 = E_{2k}$. Show that the series is absolutely and uniformly convergent for $2k \geq 4$. More interestingly, show that for $n \geq 1$ (and $2k \geq 4$) the Poincaré series is a *cuspform*. (*Hint:* Show that P_n is rapidly decreasing as $y \to +\infty$.)

Discussion: *[... iou ...]*

[07.9] Show that $\dfrac{dx\,dy}{y^2}$ gives an $SL_2(\mathbb{R})$-invariant measure on the upper half-plane $\mathfrak{H}$. Equivalently, $\dfrac{dx \wedge dy}{y^2}$ is an $SL_2(\mathbb{R})$-invariant 2-form on $\mathfrak{H}$.

Discussion: Change-of-measure constants for differentiable maps on nice spaces can be determined by computing the absolute value of the determinant of the Jacobian. Also, recall that, for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$, by direct computation $\mathrm{Im}\,(gz) = \mathrm{Im}\,(z)/|cz+d|^2$.

We should resist brute-forcing a Jacobian computation. In fact, although not completely elementary, it is standard that for a (nice) topological group G with left-and-right translation-invariant (positive, regular, Borel) measure, which $G = SL_2(\mathbb{R})$ does have, and for (for example) compact subgroup K, there is a very general theorem which implies that there is a unique left G-invariant measure on G/K such that for $f \in C^o_c(G)$, we have the obviously-desirable property

$$\int_G f \;=\; \int_{G/K} \Big(\int_K f(gk)\, dk \Big)\, dg$$

with the obvious sense to the notation. Thus, it is useful to know that $\mathfrak{H} \approx G/K$ with $G = SL_2(\mathbb{R})$ and $K = SO(2, \mathbb{R})$.

In particular, if, for a subgroup $H \subset G$, left H-invariance of a (nice) measure on G/K already uniquely determines that measure (up to scalars), that uniquely-determined thing must be G-invariant, since we know that a G-invariant thing is unique, and *exists.*

In the case at hand: by basic linear algebra, with

$$M = \{m_a = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbb{R}^\times\} \qquad N = \{n_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R}\} \qquad w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and $P = NM = MN$, we know (and it is elementary to check) the *Bruhat decomposition*

$$SL_2(\mathbb{R}) \;=\; P \cup PwN \qquad\qquad \text{(disjoint union)}$$

Since $SL_2(\mathbb{R})$ really does act on $\mathfrak{H}$ by a *group* action, meaning that we have the associativity $g(h(z)) = (gh)(z)$ for $g, h \in SL_2(\mathbb{R})$ and $z \in \mathfrak{H}$, and since Jacobians compose, it would suffice to compute the Jacobian of elements of M, N, and of w.

The element $m_a \in M$ acts just dilation by a^2, so its Jacobian at a point $z \in \mathfrak{H}$ is

$$J(m_a, z) \;=\; \begin{pmatrix} \frac{\partial ax}{\partial x} & \frac{\partial ay}{\partial x} \\ \frac{\partial ax}{\partial y} & \frac{\partial ay}{\partial y} \end{pmatrix} \;=\; \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$$

So the determinant is a^2, and is independent of z. The element n_{x_o} acts by translation by x_o, with Jacobian

$$J(n_{x_o}, z) \;=\; \begin{pmatrix} \frac{\partial x+x_o}{\partial x} & \frac{\partial y}{\partial x} \\ \frac{\partial x+x_o}{\partial y} & \frac{\partial y}{\partial y} \end{pmatrix} \;=\; \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Already this shows that any function $m(z)$ so that $m(z) \cdot dx\,dy$ is an invariant measure on $\mathfrak{H}$ is a constant multiple of y^{-2}. By the abstract uniqueness assertion, there are no others. Thus, *by pure thought,* $dx\,dy/y^2$ is $SL_2(\mathbb{R})$ invariant.

But, suppose we want to see a computation... if only to learn that we do not learn much from it, and, that it is less persuasive than the idea just discussed. That is, even after the linear algebra simplification, we need to see effect of the Weyl element w. First, there is the computation of the behavior of $x = \mathrm{Re}(z)$ and $y = \mathrm{Im}(z)$ under $z \to w(z) = -1/z$: with $z = x + iy$ as usual,

$$2\mathrm{Re}(-1/z) \;=\; \frac{-1}{x+iy} + \frac{-1}{x-iy} \;=\; \frac{-(x-iy)-(x+iy)}{|x+iy|^2} \;=\; 2\cdot\frac{-x}{x^2+y^2}$$

We should already know that $\mathrm{Im}(g(z)) = \mathrm{Im}(z)/|cz+d|^2$, but this special case is easily recomputed:

$$2i\mathrm{Im}(-1/z) \;=\; \frac{-1}{x+iy} - \frac{-1}{x-iy} \;=\; \frac{-(x-iy)+(x+iy)}{|x+iy|^2} \;=\; 2i\cdot\frac{y}{x^2+y^2}$$

I will write out this computation later... :)

The possible pointlessness of doing brittle computations is something we should understand. Sometimes there is no better option, but, if there *is* an option, it is potentially desirable. One must do the cost-benefit analysis. ///

[07.10] *(Petersson inner product)* For holomorphic level-one cuspforms f, g of weight $2k$, show that

$$z \longrightarrow f(z)\cdot\overline{g(z)}\cdot y^{2k}$$

is Γ-*invariant*, that is, the action of Γ on it does *not* involve a cocycle, in contrast to the individuals f, g, y^{2k}. Show that

$$\langle f, g\rangle_{2k} \;=\; \int_{\Gamma\backslash\mathfrak{H}} f(z)\cdot\overline{g(z)}\cdot y^{2k}\,\frac{dx\,dy}{y^2}$$

gives a hermitian inner product on weight-$2k$ level-one cuspforms. Here the domain of integration $\Gamma\backslash\mathfrak{H}$ can be taken to be the standard fundamental domain for Γ, if desired, for concreteness.

Discussion: *[... iou ...]*

[07.11] Show that for $n \geq 1$ and weight $2k \geq 4$, taking inner products with Poincaré series gives Fourier coefficients of cuspforms. That is, show that there is an explicit constant C_n such that, for every holomorphic weight-$2k$ level-one cuspform f,

$$\langle f, P_n\rangle_{2k} \;=\; C_n\cdot(n^{th}\text{ Fourier coefficient of } f)$$

It would be wise to use (prove?) the *unwinding*

$$\int_{\Gamma\backslash\mathfrak{H}} f(z)\cdot\overline{P_n(z)}\cdot y^{2k}\,\frac{dx\,dy}{y^2} \;=\; \int_{\Gamma_\infty\backslash\mathfrak{H}} f(z)\cdot\overline{\psi_n(z)}\cdot y^{2k}\,\frac{dx\,dy}{y^2}$$

and the fact that $\Gamma_\infty\backslash\mathfrak{H}$ has a very convenient set of representatives (*fundamental domain*)

$$F \;=\; \{z = x+iy : x\in[0,1],\; y\in(0,+\infty)\}$$

Then expand f in its Fourier expansion and integrate termwise.

Discussion: *[... iou ...]*

[07.12] Show that $\displaystyle\sum_{0\neq\lambda\in\mathbb{Z}[i]} \frac{1}{\lambda^4} \neq 0.$

(*Hint:* The ring $\mathbb{Z}[i]$ is Euclidean, so is a principal ideal domain, has units just $\pm1, \pm i$. Therefore, that expression has an *Euler product.*)

Discussion: For the Euler-Riemann zeta function (attached to $\mathbb{Z}$), the convergence of its Euler product

$$\zeta(s) \;=\; \sum_{n\geq 1} \frac{1}{n^s} \;=\; \prod_{p \text{ prime}} \frac{1}{1-p^{-s}} \qquad\qquad \text{(in the region } \mathrm{Re}(s) > 1)$$

proves that $\zeta(s) \neq 0$ in $\mathrm{Re}(s) > 1$. The choice to sum only over *positive* integers n is a convenient choice of representatives for non-zero integers modulo the units $\{\pm 1\}$, and this set of representatives is conveniently closed under multiplication, and contains the usual primes.

If we anticipate that at some point, for more general rings R than $\mathbb{Z}$, it might be awkward to try to find a set of representatives for R modulo units, also closed under multiplication, etc., we can dodge this problem. In the relatively docile situation where R has only finitely-many units $R^\times$, we can divide by the cardinality of $R^\times$. Already for $\mathbb{Z}$, we can illustrate this:

$$\zeta(s) \;=\; \frac{1}{\mathrm{card}\,\mathbb{Z}^\times} \sum_{n\neq 0} \frac{1}{|n|^s}$$

Analously, the ring $\mathbb{Z}[i]$ is Euclidean, with a finite number of units, just ± 1 and $\pm i$. Notably, the fourth power of each of these units is 1, so $(\lambda\cdot\eta)^4 = \lambda^4$ for all $\eta \in \mathbb{Z}[i]^\times$. Thus,

$$\sum_{0\neq\lambda\in\mathbb{Z}[i]} \frac{1}{\lambda^4} \;=\; \frac{1}{4} \sum_{\lambda\in\mathbb{Z}[i]/\mathbb{Z}[i]^\times} \frac{1}{\lambda^4}$$

Let $p_1, p_2, \ldots$ be the primes in $\mathbb{Z}[i]$, modulo units. Factor λ *uniquely* into such primes (modulo $\mathbb{Z}[i]^\times$) in $\mathbb{Z}[i]$: $\lambda = p_1^{e_1} \ldots p_n^{e_1}$ with only finitely-many primes occurring. Thus,

$$\sum_{\lambda\in\mathbb{Z}[i]/\mathbb{Z}[i]^\times} \frac{1}{\lambda^4} \;=\; \sum \frac{1}{(p_1^{e_1}\ldots p_n^{e_1})^4} \qquad\qquad \text{(summed over finite products of powers of primes)}$$

By unique factorization (as in $\mathbb{Z}$), and summing geometric series, this is

$$\prod_{p_j} \sum_{e_j\geq 0} \frac{1}{(p_j^{e_j})^4} \;=\; \prod_{p_j} \sum_{e_j\geq 0} \frac{1}{1-p_j^{-4}}$$

If we are sure that this infinite product converges, then, since no factor is 0, the product cannot be 0.

Indeed, the infinite product does converge, and the following is a fairly elegant way to see this. Taking absolute values, we also have an Euler product

$$\zeta_{\mathbb{Z}[i]}(s) \;=\; \sum_{\lambda\in\mathbb{Z}[i]/\mathbb{Z}[i]^\times} \frac{1}{|\lambda|^{2s}} \;=\; \prod_{p_j} \frac{1}{1-|p_j|^{2s}}$$

where the 2 in the exponents is both traditional and a good normalization.

Depending on one's prior knowledge, there are a few different ways to proceed from this point. One approach uses the behavior of prime integers $p \in \mathbb{Z}$ in the larger ring $\mathbb{Z}[i]$, namely, that $2 = -i(1+i)^2$ (essentially a square), that $p = 1 \bmod 4$ factors (*splits*) as a product of two primes $p = p_1p_2$ in $\mathbb{Z}[i]$ (not differing by a unit), with $p_2 = \overline{p_1}$ (complex conjugate), and $p = 3 \bmod 4$ stays prime in $\mathbb{Z}[i]$, and this accounts for all primes in $\mathbb{Z}[i]$ (up to units). For the primes $p \in \mathbb{Z}$ that split as $p = p_1p_2$, necessarily $|p_1| = |p_2| = \sqrt{p}$, since the two are complex conjugates (up to units, anyway).

This gives more details about the Euler product: grouping according to which prime in $\mathbb{Z}$ the primes in $\mathbb{Z}[i]$ divide.

$$\zeta_{\mathbb{Z}[i]}(s) \;=\; \frac{1}{1-|1+i|^{-2s}} \cdot \prod_{p=1(4)} \frac{1}{1-|p_1|^{-2s}} \frac{1}{1-|p_2|^{-2s}} \cdot \prod_{p=3(4)} \frac{1}{1-|p|^{-2s}}$$

$$= \frac{1}{1-2^{-s}} \cdot \prod_{p=1(4)} \frac{1}{1-p^{-s}} \frac{1}{1-p^{-s}} \cdot \prod_{p=3(4)} \frac{1}{1-p^{-2s}}$$

$$= \frac{1}{1-2^{-s}} \cdot \prod_{p=1(4)} \frac{1}{1-p^{-s}} \frac{1}{1-p^{-s}} \cdot \prod_{p=3(4)} \frac{1}{1-p^{-s}} \frac{1}{1+p^{-s}}$$

Direct estimates or comparison with the Euler product for $\zeta(s)$ show that this converges (absolutely...) in $\mathrm{Re}(s) > 1$. ///
