(*April 21, 2021*)

Complex analysis examples discussion 08

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/˜garrett/

[This document is http://www.math.umn.edu/˜garrett/m/complex/examples_2020-21/cx_discussion_08.pdf]

[08.1] Determine the genus of the curve $y^2 = x^5 - 1$.

Discussion: This is a *hyper-elliptic* curve, being of the form $y^2 =$ square-free polynomial in x. That $x^5 - 1$ is square-free in $\mathbb{C}[x]$ is clear in at least two ways: one way is to observe that $x^5 - 1$ has no common factors with its derivative $5x^4$. The Riemann-Hurwitz formula for the genus g of a hyper-elliptic curve of degree d simplifies:

$$2 - 2g \;=\; \begin{cases} 2 \cdot (2 - 2 \cdot 0) - d & \text{(for } d \text{ even)} \\ \\ 2 \cdot (2 - 2 \cdot 0) - (d+1) & \text{(for } d \text{ odd)} \end{cases}$$

or

$$g \;=\; \begin{cases} \dfrac{d}{2} - 1 & \text{(for } d \text{ even)} \\ \dfrac{d+1}{2} - 1 & \text{(for } d \text{ odd)} \end{cases}$$

For $d = 5$, this gives $g = \frac{5+1}{2} - 1 = 3 - 1 = 2$. ///

[08.2] Show a change of variables to convert $y^2 = x^6 - 1$ to something of the form $y^2 =$ quintic in x.

Discussion: To achieve this effect, find a linear fractional transformation g sending ∞ to one of the zeros of $x^6 - 1$, such as $x \to \frac{x+1}{x}$. Replacing x by $\frac{x+1}{x}$ in the equation gives

$$y^2 \;=\; \Big(\frac{x+1}{x}\Big)^6 - 1$$

or

$$x^6 \cdot y^2 \;=\; (x+1)^6 - x^6 \;=\; 6x^5 + 15x^4 + 20x^3 + 15x^2 + 6x + 1$$

Replacing y by y/x^3 gives

$$y^2 \;=\; 6x^5 + 15x^4 + 20x^3 + 15x^2 + 6x + 1$$

as desired. ///

[08.3] Determine the genus of the curve $y^3 = x^3 - 1$.

Discussion: This ramified covering of $\mathbb{P}^1$ by $(x, y) \to x$ is of degree 3, and there are three distinct local cube root functions y above all $x \in \mathbb{C}$ *except* the three zeros $1, \omega, \omega^2$ of $x^3 - 1$, since there is no cube root function on a neighborhood of 0. These points are *totally ramified*, so of ramification index $e = 3$ in a three-fold ramified cover.

We can also look at the Newton polygons to confirm the total ramification: the coefficients of $y^3 - (x^3 - 1)$ have vanishing order $0, \infty, \infty, 1$ at each of the three zeros, so the Newton polygons have slope 1/3, and length 3.

As in the notes, the compactification of Fermat curves in $\mathbb{P}^2$ works well. Namely, in homogeneous coordinates x, y, t, the homogenized equation is $y^3 = x^3 - t^3$. The points at infinity are at $t = 0$, and the equation is $y^3 = x^3$ (not both 0). Thus, mod $\mathbb{C}^\times$, there are 3 distinct points on the curve over the point $\infty \in \mathbb{P}^1$.

Alternatively, to use the compactification in $\mathbb{P}^1 \times \mathbb{P}^1$, use coordinates $1/x, 1/y$ in place of x, y, and look near 0: $(1/y)^3 = (1/x)^3 - 1$ simplifies to $x^3 = y^3 - x^3y^3$ or $y^3 = x^3/(1 - x^3)$. Near $x = 0$, there are 3 *distinct* cube roots of $1/(1 - x^3)$, so there are three distinct holomorphic functions $y = x/(1 - x^3)^{1/3}$, $y = \omega x/(1 - x^3)^{1/3}$, and $y = \omega^2 x/(1-x^3)^{1/3}$ near $x = 0$. That is, there is *no* ramification above ∞. It is true that the curve *self-intersects* above ∞, since those three functions y all take the same value above $x = \infty$. The self-intersection can be corrected by a suitable blow-up, as in the notes, and does not affect ramification.

By Riemann-Hurwitz, the genus g of this ramified cover is determined by

$$2 - 2g \;=\; 3 \cdot (2 - 2 \cdot 0) - \sum_{x_o = 1, \omega, \omega^2} (e_{x_o} - 1) \;=\; 6 - \sum_{x_o = 1, \omega, \omega^2} (3 - 1) \;=\; 6 - 3 \cdot 2 \;=\; 0$$

Thus, $g = 1$. ///

[08.4] Determine the genus of the curve $y^3 = x^4 - 1$.

This ramified covering of $\mathbb{P}^1$ by $(x, y) \to x$ is of degree 4, and there are three holomorphic functions y above all $x \in \mathbb{C}$ except the four (distinct) zeros $\pm 1, \pm i$ of $x^4 - 1$, since there is no cube root function on a neighborhood of 0. These four points are *totally ramified*, so of ramification index $e = 3$ in a three-fold ramified cover.

Newton polygons confirm the total ramification: the coefficients of $y^3 - (x^4 - 1)$ have vanishing order $0, \infty, \infty, 1$ at each of the four zeros, so the Newton polygons have slope $1/3$, and length 3.

To determine the ramification of the $\mathbb{P}^1 \times \mathbb{P}^1$ compactification above ∞, use coordinates $1/x, 1/y$ in place of x, y, and look near 0: $(1/y)^3 = (1/x)^4 - 1$ simplifies to $x^4 = y^3 - x^4 y^3$ or $y^3 = x^4/(1 - x^4)$. The Newton polygon of $y^3 - x^4/(1 - x^4)$ at x has vanishing orders $0, \infty, \infty, 4$, so has slope $4/3$. The *rise* and *run* are relatively prime, so ramification above ∞ is *total*: degree 3. Yes, there is self-intersection, but we can resolve the singularity.

By Riemann-Hurwitz, the genus g of this ramified cover is determined by

$$2 - 2g \;=\; 3 \cdot (2 - 2 \cdot 0) - \sum_{x_o = \pm 1, \pm i} (e_{x_o} - 1) - (e_\infty - 1) \;=\; 6 - \sum_{x_o = \pm 1, \pm i} (3 - 1) - (3 - 1) \;=\; 6 - 4 \cdot 2 - 2 \;=\; -4$$

Thus, $g = 3$. ///

[08.5] Determine the ramification above $x = 0$ in the ramified cover $(x, y) \to x \in \mathbb{P}^1$ where $y^5 + xy^2 + x^2 = 0$.

Discussion: Since the polynomial is not easily explicitly solvable for y, we use the Newton polygon of the polynomial $y^5 + xy^2 + x^2$, taking orders with respect to x: the orders of the coefficients are $0, \infty, \infty, 1, \infty, 2$. Thus, there is a length-3 segment of slope $2/3$, and a length-2 segment of slope $1/2$. Thus, there is a point with ramification index 3 (the multiplicative inverse of the slope), and another point with ramification index 2 above $x = 0$. There is also self-intersection. ///

[08.6] Determine the ramification above $x = 0$ in the ramified cover $(x, y) \to x \in \mathbb{P}^1$ where $y^5 + x^2 y^2 + x^2 = 0$.

Discussion: Use the Newton polygon of the polynomial $y^5 + x^2 y^2 + x^2$, taking orders with respect to x: the orders of the coefficients are $0, \infty, 2, \infty, 2$. Thus, the coefficient of y^2 lies above the Newton polygon, which then has a length-5 segment of slope $2/5$. The rise and run are relatively prime, so there is a single, totally ramified point over x. There is also self-intersection. ///

[08.7] Show that a ramified cover $f : E_1 \to E_2$ of elliptic curves E_j must actually be *unramified*, that is, not ramified at any point.

Discussion: By Riemann-Hurwitz, for a ramified cover of degree n of two elliptic curves,

$$(2 - 2 \cdot 1) \;=\; n \cdot (2 - 2 \cdot 1) - \sum_{\text{rfd } y} (e_y - 1)$$

That is,

$$0 = \sum_{\text{rfd } y} (e_y - 1)$$

Thus, no $e_y > 1$. ///

[08.8] Show that in a ramified cover $C_1 \to C_2$ of compact connected Riemann surfaces, the genus of C_1 must be at least the genus of C_2.

Discussion: Let g_i be the genus of C_i, and n the degree of the ramified cover. If $g_2 = 0$, certainly $g_1 \geq g_0$, so suppose $g_2 \geq 1$. Riemann-Hurwitz is

$$(2 - 2 \cdot g_1) = n \cdot (2 - 2 \cdot g_2) - \sum_{\text{rfd } y} (e_y - 1)$$

Rearranging,

$$2g_1 - 2 = n \cdot (2g_2 - 2) + \sum_{\text{rfd } y} (e_y - 1) \geq n \cdot (2g_2 - 2)$$

Using $g_2 - 1 \geq 0$,

$$g_1 \geq 1 + n \cdot (g_2 - 1) \geq 1 + 1 \cdot (g_2 - 1) = g_2$$

as claimed. ///

[08.9] Determine the points z such that there is non-trivial ramification over z in the ramified covering $(z, w) \to z$ from the curve $w^5 + 5zw + z^3 = 0$.

Discussion: Near points z_o where there are 5 distinct values roots $w_1, \ldots, w_5$ to that quintic, the distinctness of the w_i implies that $\frac{\partial}{\partial w} w^5 + 5zw + z^3 \neq 0$ does not vanish there, so by the holomorphic inverse function theorem there are five distinct holomorphic functions w of z there. Thus, there is no ramification above such z_o.

To find points z_o above which ramification is *possible*, we compute the *greatest common divisor* of $f(w) = w^5 + 5zw + z^3$ and $f'(w) = 5w^4 + 5z$ in the Euclidean ring $\mathbb{C}(z)[w]$, by the Euclidean algorithm: the first step is

$$f(w) - \frac{y}{5} \cdot f'(w) = \Big(w^5 + 5zw + z^3\Big) - \frac{y}{5} \cdot \Big(5w^4 + 5z\Big) = 4zw + z^3$$

Away from $z = 0$, we can divide $4zw + z^3$ by z, and the remainder of $f'(w)$ after division by $w + \frac{z^2}{4}$ is the value of $f'(w)$ at $w = -z^2/4$, namely $4(-z^2/4)^4 + 5z$. This is a non-zero element of $\mathbb{C}(z)$, as expected, since the original polynomial $w^5 + 5zw + z^3$ is irreducible in $\mathbb{C}[z, w] \approx \mathbb{C}[z][w]$.

However, $w^5 + 5z_o w + z_o^3 = 0$ will have multiple roots w for $z_o \in \mathbb{C}$ such that $w^5 + 5z_o w + z_o^3$ and $5w^4 + 5z_o$ have a common factor in $\mathbb{C}[w]$. The *gcd* computation above shows that unless $z_o = 0$ or $4(-z^2/4)^4 + 5z = 0$, there is no common factor. Thus, the only possible ramification is above $z = 0$ and/or the seven roots of $z^7 = -5/64$. ///

[08.10] Let $z_1, \ldots, z_n$ be points in $\mathbb{P}^1$. Determine the dimension of the space of meromorphic functions on $\mathbb{P}^1$ with poles at most at $\{z_1, \ldots, z_n\}$, counting multiplicities.

Discussion: *(This is a very special case of the Riemann-Roch theorem.)*

We can reduce to the case that none of the z_i is ∞, by dividing by z^N, where N is the multiplicity with which ∞ appears in the list. This exchanges poles at 0 with poles at 0, and is an isomorphism of vector spaces, so does not change the dimension count.

The meromorphic functions on $\mathbb{P}^1$ are rational functions $P(z)/Q(z)$, where P, Q are polynomials, and Q is not identically 0. The poles of $P(z)/Q(z)$ are at the zeros of Q, and a pole at ∞ of order $\deg P - \deg Q$ if that number is positive. Thus, if no poles at are allowed at ∞, $\deg P \le \deg Q$. Thus, rational functions with no poles at ∞ and finite poles at $z_1, \ldots, z_n$ are of the form $P(z)/(z-z_1)\ldots(z-z_n)$ with P of degree at most n. This gives $n+1$ coefficients to be chosen for P, giving an $(n+1)$-dimensional vector space. ///

[08.11] Let $\zeta_1, \ldots, \zeta_m$ and $z_1, \ldots, z_n$ be points in $\mathbb{P}^1$. Determine the dimension of the space of meromorphic functions on $\mathbb{P}^1$ with poles at most at $\{z_1, \ldots, z_n\}$, counting multiplicities, and zeros (at least) at $\zeta_1, \ldots, \zeta_m$.

Discussion: Continuing the previous argument, the functions are of the form $P(z)(z-\zeta_1)\ldots(z-\zeta_m)/(z-z_1)\ldots(z-z_n)$ with P of degree at most $n-m$. If $m > n$ this is impossible. If $m \le n$, this leaves $(n-m)+1$ coefficients to choose, giving an $(n-m)+1$-dimensional vector space of rational functions. ///

[08.12] Let $z_1, \ldots, z_n$ be points on an elliptic curve $E = \mathbb{C}/\Lambda$. Determine the dimension of the space of meromorphic functions on E with poles at most at $\{z_1, \ldots, z_n\}$, counting multiplicities.

Discussion: *(This is another special case of the Riemann-Roch theorem.)*

For such a function f, evaluating the integral $\int_\gamma f$ around a period paralellogram (indenting suitable in case poles lie on it) directly and also by residues produces the relation $\sum_{z_j} \mathrm{Res}_{z_j} f = 0$. Also, an elliptic function without poles is constant, so two elliptic functions with matching *polar parts* differ by a constant. Thus, the dimension of the space with $n > 0$ specified poles is at most n. We claim that this bound is attained for $n > 0$.

The case $n = 0$ is treated separately. For $n = 0$, such elliptic functions are *entire*, and constant, by Liouville: the dimension is 1.

For $n = 1$, for example, since the sum of the residues is 0, f cannot have a pole, so for $n = 1$, the space of such functions is still just constants, so 1-dimensional.

One approach is by direct construction of elliptic functions.

For $n \ge 2$, we can subtract multiples of *translates* of $\wp(z)^\ell$ with $0 < \ell \in \mathbb{Z}$ and $\wp'(z) \cdot \wp(z)^\ell$ with $0 \le \ell \in \mathbb{Z}$ to leave only *simple* poles. This preserves the dimension count. Thus, we can assume that the $z_1, \ldots, z_n$ are *distinct*. Further, we can translate them, if necessary, by some small amount so that no $z_j \in \Lambda$, prove existence by construction, and then translate back at the end. For complex numbers $t_1, \ldots, t_n$ with $t_1 + \ldots + t_n = 0$, we would like to sum

$$\frac{t_1}{z-(\lambda+z_1)} + \ldots + \frac{t_n}{z-(\lambda+z_n)}$$

over $\lambda \in \Lambda$, but there will be issues of convergence, as with $\wp(z)$. To understand the asymptotic behavior as a function of λ, rearrange to

$$-\frac{1}{\lambda} \cdot \Big(\frac{t_1}{1-\frac{z-z_1}{\lambda}} + \ldots + \frac{t_n}{1-\frac{z-z_n}{\lambda}}\Big)$$

$$= \; -\frac{1}{\lambda} \cdot \Big(t_1 \cdot (1 + \frac{z-z_1}{\lambda}) + \ldots + t_n \cdot (1 + \frac{z-z_n}{\lambda}) + O(\frac{1}{\lambda^2})\Big)$$

$$= \; -\frac{1}{\lambda} \cdot \Big((t_1 + \ldots + t_n)\frac{z}{\lambda} + (t_1 z_1 + \ldots + t_n z_n)\frac{1}{\lambda}\Big) + O(\frac{1}{\lambda^3}) \; = \; -\frac{1}{\lambda^2} \cdot (t_1 z_1 + \ldots + t_n z_n) + O(\frac{1}{\lambda^3})$$

To make this $O(1/\lambda^3)$, similar to what was done with $\wp(z)$, add $\frac{1}{\lambda^2}(t_1 z_1 + \ldots + t_n z_n$, and form

$$f(z) \; = \; \sum_{\lambda \in \Lambda} \Big(\frac{t_1}{z-(\lambda+z_1)} + \ldots + \frac{t_n}{z-(\lambda+z_n)} + \frac{t_1 z_1 + \ldots + t_n z_n}{\lambda^2}\Big)$$

Thus, we have an $(n-1)$-dimensional space of elliptic functions with simple poles at the $z_1, \ldots, z_n$. ///
