Determine the genus of the curve $y^2 = x^5 - 1$.

**Discussion:** This is a hyper-elliptic curve, being of the form $y^2 = \text{square-free polynomial in } x$. That $x^5 - 1$ is square-free in $\mathbb{C}[x]$ is clear in at least two ways: one way is to observe that $x^5 - 1$ has no common factors with its derivative $5x^4$. The Riemann-Hurwitz formula for the genus $g$ of a hyper-elliptic curve of degree $d$ simplifies:

$$2 - 2g = \begin{cases} 2 \cdot (2 - 2 \cdot 0) - d & \text{for } d \text{ even} \\ 2 \cdot (2 - 2 \cdot 0) - (d + 1) & \text{for } d \text{ odd} \end{cases}$$

or

$$g = \begin{cases} \frac{d}{2} - 1 & \text{for } d \text{ even} \\ \frac{d + 1}{2} - 1 & \text{for } d \text{ odd} \end{cases}$$

For $d = 5$, this gives $g = \frac{5 + 1}{2} - 1 = 3 - 1 = 2$. ///

Show a change of variables to convert $y^2 = x^6 - 1$ to something of the form $y^2 = \text{quintic in } x$.

**Discussion:** To achieve this effect, find a linear fractional transformation $g$ sending $\infty$ to one of the zeros of $x^6 - 1$, such as $x \to \frac{x+1}{2}$. Replacing $x$ by $\frac{x+1}{2}$ in the equation gives

$$y^2 = \left(\frac{x+1}{x}\right)^6 - 1$$

or

$$x^6 \cdot y^2 = (x+1)^6 - x^6 = 6x^5 + 15x^4 + 20x^3 + 15x^2 + 6x + 1$$

Replacing $y$ by $y/x^3$ gives

$$y^2 = 6x^5 + 15x^4 + 20x^3 + 15x^2 + 6x + 1$$

as desired. ///

Determine the genus of the curve $y^3 = x^3 - 1$.

**Discussion:** This ramified covering of $\mathbb{P}^1$ by $(x, y) \to x$ is of degree 3, and there are three distinct local cube root functions $y$ above all $x \in \mathbb{C}$ except the three zeros 1, $\omega$, $\omega^2$ of $x^3 - 1$, since there is no cube root function on a neighborhood of 0. These points are totally ramified, so of ramification index $e = 3$ in a three-fold ramified cover.

We can also look at the Newton polygons to confirm the total ramification: the coefficients of $y^3 - (x^3 - 1)$ have vanishing order 0, $\infty$, $\infty$, 1 at each of the three zeros, so the Newton polygons have slope $1/3$, and length 3.

As in the notes, the compactification of Fermat curves in $\mathbb{P}^2$ works well. Namely, in homogeneous coordinates $x, y, t$, the homogenized equation is $y^3 = x^3 - t^3$. The points at infinity are at $t = 0$, and the equation is $y^3 = x^3$ (not both 0). Thus, mod $\mathbb{C}^\times$, there are 3 distinct points on the curve over the point $\infty \in \mathbb{P}^1$.

Alternatively, to use the compactification in $\mathbb{P}^1 \times \mathbb{P}^1$, use coordinates $1/x, 1/y$ in place of $x, y$, and look near 0: $(1/y)^3 = (1/x)^3 - 1$ simplifies to $x^3 = y^3 - x^3 y^3$ or $y^3 = x^3/(1 - x^3)$. Near $x = 0$, there are 3 distinct cube roots of $1/(1 - x^3)$, so there are three distinct holomorphic functions $y = x/(1 - x^3)^{1/3}$, $y = \omega x/(1 - x^3)^{1/3}$, $y = \omega^2 x/(1 - x^3)^{1/3}$.  

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and \( y = \omega^2 x/(1-x^3)^{1/3} \) near \( x = 0 \). That is, there is no ramification above \( \infty \). It is true that the curve self-intersects above \( \infty \), since those three functions \( y \) all take the same value above \( x = \infty \). The self-intersection can be corrected by a suitable blow-up, as in the notes, and does not affect ramification.

By Riemann-Hurwitz, the genus \( g \) of this ramified cover is determined by

\[
2 - 2g = 3 \cdot (2 - 2 \cdot 0) - \sum_{x_o = 1, \omega, \omega^2} (e_{x_o} - 1) = 6 - \sum_{x_o = 1, \omega, \omega^2} (3 - 1) = 6 - 3 \cdot 2 = 0
\]

Thus, \( g = 1 \). ///

\[08.4\] Determine the genus of the curve \( y^3 = x^4 - 1 \).

This ramified covering of \( \mathbb{P}^1 \) by \((x, y) \to x\) is of degree 4, and there are three holomorphic functions \( y \) above all \( x \in \mathbb{C} \) except the four (distinct) zeros \( \pm 1, \pm i \) of \( x^4 - 1 \), since there is no cube root function on a neighborhood of 0. These four zeros are totally ramified, so of ramification index \( e = 3 \) in a three-fold ramified cover.

Newton polygons confirm the total ramification: the coefficients of \( y^3 - (x^4 - 1) \) have vanishing order 0, \( \infty \), \( \infty \), 1 at each of the four zeros, so the Newton polygons have slope 1/3, and length 3.

To determine the ramification of the \( \mathbb{P}^1 \times \mathbb{P}^1 \) compactification above \( \infty \), use coordinates \( 1/x, 1/y \) in place of \( x, y \), and look near 0: \((1/y)^3 = (1/x)^4 - 1 \) simplifies to \( x^4 = y^3 - x^4 y^3 \) or \( y^3 = x^4/(1-x^4) \). The Newton polygon of \( y^3 - x^4/(1-x^4) \) at \( x \) has vanishing orders \( 0, \infty, \infty, 4 \), so has slope 4/3. The rise and run are relatively prime, so ramification above \( \infty \) is total degree 3. Yes, there is self-intersection, but we can resolve the singularity.

By Riemann-Hurwitz, the genus \( g \) of this ramified cover is determined by

\[
2 - 2g = 3 \cdot (2 - 2 \cdot 0) - \sum_{x_o = \pm 1, \pm i} (e_{x_o} - 1) - (e_{\infty} - 1) = 6 - \sum_{x_o = \pm 1, \pm i} (3 - 1) - (3 - 1) = 6 - 4 \cdot 2 - 2 = -4
\]

Thus, \( g = 3 \). ///

\[08.5\] Determine the ramification above \( x = 0 \) in the ramified cover \((x, y) \to x \in \mathbb{P}^1 \) where \( y^5 + xy^2 + x^2 = 0 \).

**Discussion:** Since the polynomial is not easily explicitly solvable for \( y \), we use the Newton polygon of the polynomial \( y^5 + xy^2 + x^2 \), taking orders with respect to \( x \): the orders of the coefficients are 0, \( \infty \), \( \infty \), 1, \( \infty \), 2. Thus, there is a length-3 segment of slope 2/3, and a length-2 segment of slope 1/2. Thus, there is a point with ramification index 3 (the multiplicative inverse of the slope), and another point with ramification index 2 above \( x = 0 \). There is also self-intersection. ///

\[08.6\] Determine the ramification above \( x = 0 \) in the ramified cover \((x, y) \to x \in \mathbb{P}^1 \) where \( y^5 + x^2y^2 + x^2 = 0 \).

**Discussion:** Use the Newton polygon of the polynomial \( y^5 + x^2y^2 + x^2 \), taking orders with respect to \( x \): the orders of the coefficients are 0, \( \infty \), 2, \( \infty \), 2. Thus, the coefficient of \( y^2 \) lies above the Newton polygon, which then has a length-5 segment of slope 2/5. The rise and run are relatively prime, so there is a single, totally ramified point over \( x \). There is also self-intersection. ///

\[08.7\] Show that a ramified cover \( f : E_1 \to E_2 \) of elliptic curves \( E_j \) must actually be unramified, that is, not ramified at any point.

**Discussion:** By Riemann-Hurwitz, for a ramified cover of degree \( n \) of two elliptic curves,

\[
(2 - 2 \cdot 1) = n \cdot (2 - 2 \cdot 1) - \sum_{r ft \_y} (e_{y} - 1)
\]
That is,
\[ 0 = \sum_{\text{rfd } y} (e_y - 1) \]

Thus, no \( e_y > 1 \).

[08.8] Show that in a ramified cover \( C_1 \to C_2 \) of compact connected Riemann surfaces, the genus of \( C_1 \) must be at least the genus of \( C_2 \).

Discussion: Let \( g_i \) be the genus of \( C_i \), and \( n \) the degree of the ramified cover. If \( g_2 = 0 \), certainly \( g_1 \geq g_0 \), so suppose \( g_2 \geq 1 \). Riemann-Hurwitz is
\[ (2 - 2 \cdot g_1) = n \cdot (2 - 2 \cdot g_2) - \sum_{\text{rfd } y} (e_y - 1) \]

Rearranging,
\[ 2g_1 - 2 = n \cdot (2g_2 - 2) + \sum_{\text{rfd } y} (e_y - 1) \geq n \cdot (2g_2 - 2) \]

Using \( g_2 - 1 \geq 0 \),
\[ g_1 \geq 1 + n \cdot (g_2 - 1) \geq 1 + 1 \cdot (g_2 - 1) = g_2 \]
as claimed.

[08.9] Determine the points \( z \) such that there is non-trivial ramification over \( z \) in the ramified covering \((z, w) \to z\) from the curve \( w^5 + 5zw + z^3 = 0 \).

Discussion: Near points \( z_o \) where there are 5 distinct values roots \( w_1, \ldots, w_5 \) to that quintic, the distinctness of the \( w_i \) implies that \( \frac{\partial}{\partial w} w^5 + 5zw + z^3 \neq 0 \) does not vanish there, so by the holomorphic inverse function theorem there are five distinct holomorphic functions \( w \) of \( z \). Thus, there is no ramification above such \( z_o \).

To find points \( z_o \) above which ramification is possible, we compute the greatest common divisor of \( f(w) = w^5 + 5zw + z^3 \) and \( f'(w) = 5w^4 + 5z \) in the Euclidean ring \( \mathbb{C}(z)[w] \), by the Euclidean algorithm: the first step is
\[ f(w) - \frac{y}{5} \cdot f'(w) = \left( w^5 + 5zw + z^3 \right) - \frac{y}{5} \cdot \left( 5w^4 + 5z \right) = 4zw + z^3 \]

Away from \( z = 0 \), we can divide \( 4zw + z^3 \) by \( z \), and the remainder of \( f'(w) \) after division by \( w + \frac{z^2}{4} \) is the value of \( f'(w) \) at \( w = -z^2/4 \), namely \( 4(-z^2/4)^4 + 5z \). This is a non-zero element of \( \mathbb{C}(z) \), as expected, since the original polynomial \( w^5 + 5zw + z^3 \) is irreducible in \( \mathbb{C}[z, w] \equiv \mathbb{C}[z][w] \).

However, \( w^5 + 5zw + z^3 = 0 \) will have multiple roots \( w \) for \( z_o \in \mathbb{C} \) such that \( w^5 + 5zw + z^3 \) and \( 5w^4 + 5z \) have a common factor in \( \mathbb{C}[w] \). The \( \gcd \) computation above shows that unless \( z_o = 0 \) or \( 4(-z^2/4)^4 + 5z = 0 \), there is no common factor. Thus, the only possible ramification is above \( z = 0 \) and/or the seven roots of \( z^7 = -5/64 \).

[08.10] Let \( z_1, \ldots, z_n \) be points in \( \mathbb{P}^1 \). Determine the dimension of the space of meromorphic functions on \( \mathbb{P}^1 \) with poles at most at \( \{z_1, \ldots, z_n\} \), counting multiplicities.

Discussion: (This is a very special case of the Riemann-Roch theorem.)

We can reduce to the case that none of the \( z_i \) is \( \infty \), by dividing by \( z^N \), where \( N \) is the multiplicity with which \( \infty \) appears in the list. This exchanges poles at 0 with poles at 0, and is an isomorphism of vector spaces, so does not change the dimension count.

The meromorphic functions on \( \mathbb{P}^1 \) are rational functions \( P(z)/Q(z) \), where \( P, Q \) are polynomials, and \( Q \) is not identically 0. The poles of \( P(z)/Q(z) \) are at the zeros of \( Q \), and a pole at \( \infty \) of order \( \deg P - \deg Q \) if
that number is positive. Thus, if no poles at are allowed at \( \infty \), \( \deg P \leq \deg Q \). Thus, rational functions with no poles at \( \infty \) and finite poles at \( z_1, \ldots, z_n \) are of the form \( P(z)/(z - z_1) \ldots (z - z_n) \) with \( P \) of degree at most \( n \). This gives \( n + 1 \) coefficients to be chosen for \( P \), giving an \((n+1)\)-dimensional vector space.  

[08.11] Let \( \zeta_1, \ldots, \zeta_m \) and \( z_1, \ldots, z_n \) be points in \( \mathbb{P}^1 \). Determine the dimension of the space of meromorphic functions on \( \mathbb{P}^1 \) with poles at most at \( \{z_1, \ldots, z_n\} \), counting multiplicities, and zeros (at least) at \( \zeta_1, \ldots, \zeta_m \).

**Discussion:** Continuing the previous argument, the functions are of the form \( P(z)/(z - \zeta_1) \ldots (z - \zeta_m)/(z - z_1) \ldots (z - z_n) \) with \( P \) of degree at most \( n - m \). If \( m > n \) this is impossible. If \( m \leq n \), this leaves \((n-m)+1\) coefficients to choose, giving an \((n-m)+1\)-dimensional vector space of rational functions.

[08.12] Let \( z_1, \ldots, z_n \) be points on an elliptic curve \( E = \mathbb{C}/\Lambda \). Determine the dimension of the space of meromorphic functions on \( E \) with poles at most at \( \{z_1, \ldots, z_n\} \), counting multiplicities.

**Discussion:** (This is another special case of the Riemann-Roch theorem.)

For such a function \( f \), evaluating the integral \( \int \gamma f \) around a period parallelogram (indenting suitable in case poles lie on it) directly and also by residues produces the relation \( \sum z_j \text{ Res}_z f = 0 \). Also, an elliptic function without poles is constant, so two elliptic functions with matching polar parts differ by a constant. Thus, the dimension of the space with \( n > 0 \) specified poles is at most \( n \). We claim that this bound is attained for \( n > 0 \).

The case \( n = 0 \) is treated separately. For \( n = 0 \), such elliptic functions are *entire*, and constant, by Liouville: the dimension is 1.

For \( n = 1 \), for example, since the sum of the residues is 0, \( f \) cannot have a pole, so for \( n = 1 \), the space of such functions is still just constants, so 1-dimensional.

One approach is by direct construction of elliptic functions.

For \( n \geq 2 \), we can subtract multiples of translates of \( \varphi(z) \) with \( 0 < \ell \in \mathbb{Z} \) and \( \varphi(z) \cdot \varphi(z)^\ell \) with \( 0 \leq \ell \in \mathbb{Z} \) to leave only *simple* poles. This preserves the dimension count. Thus, we can assume that the \( z_1, \ldots, z_n \) are *distinct*. Further, we can translate them, if necessary, by some small amount so that no \( z_j \in \Lambda \), prove existence by construction, and then translate back at the end. For complex numbers \( t_1, \ldots, t_n \) with \( t_1 + \ldots + t_n = 0 \), we would like to sum

\[
\frac{t_1}{z - (\lambda + z_1)} + \ldots + \frac{t_n}{z - (\lambda + z_n)}
\]

over \( \lambda \in \Lambda \), but there will be issues of convergence, as with \( \varphi(z) \). To understand the asymptotic behavior as a function of \( \lambda \), rearrange to

\[
-\frac{1}{\lambda} \cdot \left( \frac{t_1}{1 - \frac{z - z_1}{\lambda}} + \ldots + \frac{t_n}{1 - \frac{z - z_n}{\lambda}} \right) = -\frac{1}{\lambda} \cdot \left( t_1 \cdot \left( 1 + \frac{z - z_1}{\lambda} \right) + \ldots + t_n \cdot \left( 1 + \frac{z - z_n}{\lambda} \right) + O\left( \frac{1}{\lambda^2} \right) \right)
\]

\[
= -\frac{1}{\lambda} \cdot \left( t_1 \cdot \left( t_1 z_1 + \ldots + t_n z_n \right) \frac{1}{\lambda} \right) + O\left( \frac{1}{\lambda^3} \right) = -\frac{1}{\lambda^3} \cdot \left( t_1 z_1 + \ldots + t_n z_n \right) + O\left( \frac{1}{\lambda^3} \right)
\]

To make this \( O(1/\lambda^3) \), similar to what was done with \( \varphi(z) \), add \( \frac{1}{\lambda^3} (t_1 z_1 + \ldots + t_n z_n) \), and form

\[
f(z) = \sum_{\lambda \in \Lambda} \left( \frac{t_1}{z - (\lambda + z_1)} + \ldots + \frac{t_n}{z - (\lambda + z_n)} + \frac{t_1 z_1 + \ldots + t_n z_n}{\lambda^2} \right)
\]

Thus, we have an \((n-1)\)-dimensional space of elliptic functions with simple poles at the \( z_1, \ldots, z_n \).  

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\(4\)