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Complex analysis examples 02

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[This document is http://www.math.umn.edu/~garrett/m/complex/examples_2020-21/cx_ex_02.pdf]

If you want feedback from me on your treatment of these examples, please email your work to me by Friday, Oct 16.

[02.1] Parametrize counter-clockwise a circle γ of radius $r > 0$ centered at z_o, and *directly* compute $\int_\gamma (z - z_o)^n \, dz$ for all positive and negative integers n.

[02.2] Of what rational function is $\sum_{n=0}^\infty n(n-1)(n-2)(n-3)z^n$ the power series expansion at 0?

[02.3] Determine the Laurent expansions of $\dfrac{1}{1-z}$ in $|z| < 1$, and in $|z| > 1$. Observe that these two have no common region of convergence.

[02.4] Using only geometric series expansions, determine the Laurent expansion of $f(z) = 1/(z-1)(z-2)$ in the annulus $1 < |z| < 2$, and also in the annulus $|z| > 2$.

[02.5] Determine the Laurent expansion of $f(z) = 1/(z-1)^3$ in the annulus $|z| > 1$, and in the annulus $|z - 1| > 0$.

[02.6] Show that an entire function f satisfying $|f(z)| \le C \cdot (1 + |z|)^{1/2}$ for some constant C is *constant*.

[02.7] Show that an entire function f satisfying $|f(z)| \le C \cdot (1 + |z|)^r$ for some $0 \le r \in \mathbb{R}$, and for some constant C, is a polynomial of degree at most r. (Yes, degrees of not-identically-zero polynomials are non-negative integers.)

[02.8] Show that an entire function f satisfying $|f(z)| \le C \cdot \log(1 + |z|)$ for some constant C is *constant*.

[02.9] Compute $\displaystyle\int_{-\infty}^\infty \frac{dx}{x^2+1}$ without using arctangent.

[02.10] Compute $\displaystyle\int_{-\infty}^\infty \frac{dx}{x^4+1}$

[02.11] Compute $\displaystyle\int_{-\infty}^\infty \frac{x\, dx}{x^4+1}$

[02.12] Compute $\displaystyle\int_{-\infty}^\infty \frac{x^2\, dx}{x^4+1}$

[02.13] Compute $\displaystyle\int_{-\infty}^\infty \frac{e^{itx}\, dx}{x^2+1}$ with real t.

[02.14] Compute $\displaystyle\int_{-\infty}^\infty \frac{\sin(tx)\, dx}{x^2+1}$ with real t.

[02.15] Compute $\displaystyle\int_{-\infty}^\infty \frac{\cos(tx)\, dx}{x^2+1}$ with real t.

[02.16] Compute $\int_{-\infty}^{\infty} \frac{e^{itx}\,dx}{(x+i)^2}$ with real t.

[02.17] Compute $\int_{-\infty}^{\infty} \frac{e^{itx}\,dx}{(x+i)^{100}}$ with real t.

[02.18] Compute $\int_0^{\infty} \frac{x\,dx}{1+x^3}$

[02.19] Compute $\int_0^{\infty} \frac{x^{1/4}\,dx}{1+x^2}$

[02.20] Compute $\frac{1}{1} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \ldots$

[02.21] Show that $z^n + z - 1$ has n zeros inside the circle $|z| = 2$.

[02.22] Are there complex zeros of $z - \cos z$ beyond the obvious one on $\mathbb{R}$?

[02.23] For a bounded sequence of complex numbers c_n, prove that $\sum_{n=0}^{\infty} c_n \frac{z^n}{z^n+1}$ converges to a holomorphic function on $|z| < 1$.

[02.24] Let f be a continuous, bounded real-valued function on $\mathbb{R}$, extending to a bounded, holomorphic function on the upper half-plane $\mathfrak{H}$. Show f is constant.

[02.25] Evaluate the Fourier transform $\int_{-\infty}^{\infty} e^{-itx} \cdot \frac{1}{(x+i)^s}\,dx$ for complex s with $\mathrm{Re}(s) > 1$, using the Gamma function.

[02.26] Show that $f(z) = \int_0^1 \frac{dt}{t \cdot z + (1-t) \cdot z_o}$ is holomorphic at any z_1 such that 0 is *not* on the straight line segment with endpoints z_o and z_1. Find the radius of convergence of its power series expanded at $z_o = -4 + 3i$.
