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Complex analysis examples 04

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[This document is http://www.math.umn.edu/~garrett/m/complex/examples_2020-21/cx_ex_04.pdf]

If you want feedback from me on your treatment of these examples, please email your work to me by Friday, Dec 04.

[04.1] Prove that, given holomorphic f, g on a non-empty open set U, and given a *simple* zero z_o of f, for all small-enough complex ε there is a unique zero of $f + \varepsilon g$ nearest z_o and it is *simple*.

[04.2] For small $w \in \mathbb{C}$, let $f(w)$ be the *simple* zero of $z^5 - z + w = 0$ near 0. Determine a few terms of the power series expansion of $f(w)$ at $w = 0$.

[04.3] Exhibit a linear fractional transformation mapping $1, 2, 3$ to z_1, z_2, z_3.

[04.4] Exhibit a linear fractional transformation mapping the circle $|z| = 1$, minus a point, to the line $\mathrm{Re}(z) = \mathrm{Im}(z)$.

[04.5] Let z, z' be points in the open upper half-plane $\mathfrak{H}$. Exhibit a linear fractional transformation stabilizing $\mathfrak{H}$ and mapping z to z'.

[04.6] Let z, z' be points in the open unit disk $\mathfrak{D}$. Exhibit a linear fractional transformation stabilizing $\mathfrak{D}$ and mapping z to z'.

[04.7] Exhibit a conformal map of the open half disk $\{z : |z| < 1, \mathrm{Re}(z) > 0\}$ to the open unit disk.

[04.8] Exhibit a conformal map of the open unit disk with $[0, 1)$ removed to the open unit disk.

[04.9] Exhibit a conformal map of the sector $\{re^{i\theta} : r > 0,\ 0 < \theta < \frac{\pi}{4}\}$ to the open unit disk.

[04.10] Exhibit a conformal map from the strip $\{z = x + iy : c < ax + by < c'\}$ to the crescent

$$\Omega \;=\; \{z : |z| < 1,\ |z - \tfrac{1}{2}| > \tfrac{1}{2}\}$$

[04.11] Let f be holomorphic on $\mathbb{C}$, and meromorphic at infinity, with a pole of order N. Show that f is a polynomial of degree N (and conversely).

[04.12] Let holomorphic $f : \mathbb{CP}^1 \to \mathbb{CP}^1$ be 2-to-1. Show that there are two linear fractional transformations α, β such that $\alpha \circ f \circ \beta$ is the map $z \to z^2$.
